1.) Consider a general two-level system with a Hamiltonian

$$\hat{H}_0 = \begin{pmatrix} E_0^0 & 0 \\ 0 & E_1^0 \end{pmatrix} \quad \hat{V} = \begin{pmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{pmatrix} \quad E_0^0 \leq E_1^0 \quad V_{01} = V_{10}$$

a.) Find the exact eigenvalues of $\hat{H} = \hat{H}_0 + \hat{V}$

b.) Expand both energies to order V^2

c.) <u>No crossing theorem</u> - As V increases, one might expect the lower level to become higher than the higher level. Show this cannot occur if $V_{01} \neq 0$.

d.) Specialize to the degenerate case $E_0^0 = E_1^0$. Can you expand the energies for small V? To what order in V are the corrections to E_0?

e.) Now consider also $V_{00} = V_{11} = 0$. Compare the exact result to an incorrect application of nondegenerate Rayleigh-Schrödinger perturbation theory to first order. Compare to Wigner-Brillouin pert theory at second order.

2.) In the case where $\hat{P}_k \hat{V} \hat{P}_k = \hat{V}_k = 0$, the degeneracy is not lifted at all to first order in deg PT. Then all $E^{(1)}_{k,nn} = 0$ and $|k,n_k\rangle^{(1)}_{11}$ are undetermined.

a.) Show that second order PT gives another eigenvalue problem for $|k, n_k\rangle_{11}^{(2)}$.

b.) Show the equation for the energy shifts is

$$\det\left(V_k - E_k^{(2)}\right) = 0 \qquad \text{with}$$

$$V^k_{ij} = \sum_{k' n_k'} \frac{{}_0\langle k i | \hat{V} | k' n_k' \rangle_0 \; {}_0\langle k' n_k' | \hat{V} | k j \rangle_0}{E^0_k - E^0_{k'}}$$

known as the matrix of second order matrix elements.

c.) Consider the Hamiltonian

$$H_0 = \begin{pmatrix} E_0^0 & 0 & 0 \\ 0 & E_0^0 & 0 \\ 0 & 0 & E_1^0 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a^* & b^* & 0 \end{pmatrix}$$

Use pert. theory to compute all three energies to $O(V^2)$ and determine $|k, n_k\rangle_{11}^{(2)}$ for the 2×2 degenerate space.

d.) Compute the energies of the Hamiltonian in c exactly. Compare the energies to the pert. expansion by plotting on the same plot. Choose $E_0^0 = 0$ $E_1^0 = 1$ and

$a = b$ $\quad 0 < a < 5$ $\quad$ and $\quad a = \frac{1}{2} b$ $\quad 0 < a < 2.5$

3.) Hyperfine splitting

We want to consider the effect of nuclear spins on the atomic energy levels. In particular, we will work with Hydrogen, taking into account that the nucleus is a proton with spin $\frac{1}{2}$ and has a magnetic moment.

Let $\hat{\vec{s}}$ be the electron spin operator and $\hat{\vec{I}}$ be the proton spin operator. The magnetic moment of the proton is

$$\vec{m}_p = g_p \frac{e\hbar}{2Mc} \vec{I} \qquad M = \text{mass of proton} \qquad g_p = 5.56$$

and of the electron is

$$\vec{m}_e = -g_e \frac{e\hbar}{2mc} \vec{s} \qquad m = \text{mass of electron} \qquad g_e = 2$$

If we start analyzing classically, we imagine the proton moment produces a magnetic field

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{A} = -\vec{m}_p \times \vec{\nabla} \frac{1}{r}$$

The electron experiences an interaction with this field

$$\hat{H}' = -\vec{m}_e \cdot \vec{B} = \vec{m}_e \cdot \vec{m}_p \, \nabla^2 \frac{1}{r} + (\vec{m}_e \cdot \vec{\nabla})(\vec{m}_p \cdot \vec{\nabla}) \frac{1}{r}$$

or

$$\hat{H}' = \frac{e^2 g_e g_p \hbar^2}{4mMc^2} \left\{ -\vec{s} \cdot \vec{I} \, \nabla^2 \frac{1}{r} + \vec{s} \cdot \vec{\nabla} \, \vec{I} \cdot \vec{\nabla} \frac{1}{r} \right\}$$

We will treat this to first order as a perturbation of the $l=0$ levels of Hydrogen (ignore the fine structure). For $l=0$ wave functions, the spatial wave function is symmetric, so

$$\int d\Omega \, \vec{s} \cdot \vec{\nabla} \, \vec{I} \cdot \vec{\nabla} \frac{1}{r} = \frac{\vec{s} \cdot \vec{I}}{3} \nabla^2 \frac{1}{r} \int d\Omega$$

So we write

$$\hat{H}' = -\frac{2}{3}\frac{e^2 g_e g_p \hbar^2}{4 m M c^2} \nabla^2 \frac{1}{r} \vec{S}\cdot\vec{I}$$

But $\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r})$

so $$\hat{H}' = \frac{4\pi}{3}\frac{e^2 g_e g_p \hbar^2}{2 m M c^2} \delta(\vec{r})\, \vec{S}\cdot\vec{I}$$

In the absence of the perturbation, the $l=0$ states are fourfold degenerate (electron spins $\uparrow$ and $\downarrow$, proton spins $\uparrow$ and $\downarrow$)

Work on the ground state with $n=1$ only.

The perturbation will split the levels into two different levels.

Compute the energy separation of these levels to lowest order in $\hat{H}'$.

Express your answer analytically in terms of

e, g_e, g_p, m, M, c, etc.

Numerically determine the wavelength (in cm) of a photon with energy ΔE.

This photon is very important in radio astronomy.