Last time we derived the Lippman-Schwinger equation

$$|\psi\rangle = |\psi_0\rangle + \frac{1}{E-\hat{H}_0+i\delta}\hat{V}|\psi\rangle$$

Let's examine in coordinate space by multiplying by $\langle x|$ on the left and introducing $\int |x'\rangle\langle x'| = \mathbb{I}$ between the fraction and the $\hat{V}$.

$$\psi(x) = \psi_0(x) + \int dx' \, \langle x| \frac{1}{E-\hat{H}_0+i\delta} |x'\rangle V(x')\psi(x')$$

Introduce the complete set of states $\int |p\rangle\langle p| \, dp = \mathbb{I}$

of momentum to get

$$\langle x| \frac{1}{E-\hat{H}_0+i\delta} |x'\rangle = \int dp \, \langle x|p\rangle\langle p| \frac{1}{E-\hat{H}_0+i\delta} |x'\rangle$$

But $\langle x|p\rangle = \dfrac{e^{ixp/\hbar}}{\sqrt{2\pi\hbar}}$ and $\hat{H}_0|p\rangle = \underset{\overset{\uparrow}{\text{number}}}{\dfrac{p^2}{2m}}|p\rangle$

$$\langle x| \frac{1}{E-\hat{H}_0+i\delta} |x'\rangle = \int dp \, \frac{e^{ixp/\hbar} \, e^{-ix'p/\hbar}}{(2\pi\hbar)\left(E-\frac{p^2}{2m}+i\delta\right)}$$

$$= \int dp \, \frac{e^{ip(x-x')/\hbar}}{2\pi\hbar} \frac{1}{\left(\sqrt{E+i\delta}-\frac{p}{\sqrt{2m}}\right)\left(\sqrt{E+i\delta}+\frac{p}{\sqrt{2m}}\right)}$$

Can integrate by residues. If you don't know how to do this, don't worry.

The answer is

$$\langle x| \frac{1}{E-\hat{H}_0 + i\delta} |x'\rangle = -\frac{i}{\hbar}\frac{\sqrt{2m}}{2\sqrt{E}}\left[+\Theta(x-x')\, e^{i\frac{\sqrt{2mE}}{\hbar}(x-x')} + \Theta(x'-x)\, e^{-i\frac{\sqrt{2mE}}{\hbar}(x-x')}\right]$$

$$= -i\frac{\sqrt{2mE}}{\hbar}\frac{1}{2E}\left[\Theta(x-x')\, e^{i\sqrt{2mE}(x-x')/\hbar} + \Theta(x'-x)\, e^{-i\sqrt{2mE}(x-x')/\hbar}\right]$$

So, if we let $k = \frac{\sqrt{2mE}}{\hbar}$ and choose |

$\psi_0(x) = e^{ik\cdot x}$ ⇒ $\overset{\text{incident}}{\text{wave}}$ moving to right

$$\psi(x) = e^{ik\cdot x} - \frac{ik}{2E}\int_{-\infty}^{x} e^{ik(x-x')} V(x')\psi(x')\,dx' - \frac{ik}{2E}\int_{x}^{\infty} e^{-ik(x-x')} V(x')\psi(x')\,dx'$$

$$= e^{ikx}\left(1 - \frac{ik}{2E}\int_{-\infty}^{x} e^{-ikx'} V(x')\psi(x')\,dx'\right)$$

$$- \frac{ik}{2E} e^{-ikx}\int_{x}^{\infty} e^{ikx'} V(x')\psi(x')\,dx'$$

look at limit $x \to +\infty$ ⇒ $t = 1 - \frac{ik}{2E}\int_{-\infty}^{+\infty} e^{-ikx'} V(x')\psi(x')\,dx'$

look at limit $x \to -\infty$ ⇒ $r = \frac{-ik}{2E}\int_{-\infty}^{\infty} e^{ikx'} V(x')\psi(x')\,dx'$

since ψ is scattering fcn ~~from~~ to right we write $|\psi\rangle = |\psi_\rightarrow\rangle$

$\psi_0(x) = e^{ikx} = \langle x|\psi_{0\rightarrow}\rangle$ $e^{-ikx} = \langle x|\psi_{0\leftarrow}\rangle$

so $r_\rightarrow = \frac{-ik}{2E}\langle\psi_{0\leftarrow}|\hat{V}|\psi_\rightarrow\rangle$ $t_\rightarrow = 1 - \frac{ik}{2E}\langle\psi_{0\rightarrow}|\hat{V}|\psi_\rightarrow\rangle$

Formal solution and the Born Series

$$|\psi\rangle = |\psi_0\rangle + \frac{1}{E-\hat{H}_0+i\delta}\hat{V}|\psi\rangle$$

call $\frac{1}{E-\hat{H}_0+i\delta} = \hat{G}_{0+}(E)$

$$|\psi\rangle = \left[1 - \hat{G}_{0+}(E)\hat{V}\right]^{-1}|\psi_0\rangle$$

$\hat{U}$ not in scanned notes

so $r_{\rightarrow} = \frac{-ik}{2E}\langle\psi_{0\leftarrow}|\hat{V}\left[1-\hat{G}_{0+}(E)\hat{V}\right]^{-1}|\psi_{0\rightarrow}\rangle$

$$t_{\rightarrow} = 1 - \frac{ik}{2E}\langle\psi_{0\rightarrow}|\hat{U}\left[1-\hat{G}_{0+}(E)\hat{V}\right]^{-1}|\psi_{0\rightarrow}\rangle$$

expand in a geometric series

$$|\psi_{\rightarrow}\rangle = \sum_{n=0}^{\infty}\left(\hat{G}_{0+}(E)\hat{V}\right)^n|\psi_{0\rightarrow}\rangle \qquad n\text{th term}$$

$$r_{\rightarrow} = \frac{-ik}{2E}\sum_{n=0}^{\infty}\langle\psi_{0\leftarrow}|\hat{V}\left(\hat{G}_{0+}(E)\hat{V}\right)^n|\psi_{0\rightarrow}\rangle \qquad (n+1)\text{st term}$$

$$t_{\rightarrow} = 1 - \frac{ik}{2E}\sum_{n=0}^{\infty}\langle\psi_{0\rightarrow}|\hat{V}\left(\hat{G}_{0+}(E)\hat{V}\right)^n|\psi_{0\rightarrow}\rangle \qquad (n+1)\text{st term}$$

$n=1$ is called the Born approximation

$$\psi^{Born}_{\rightarrow} = |\psi_{0\rightarrow}\rangle + \hat{G}_{0+}(E)\hat{V}|\psi_{0\rightarrow}\rangle$$

$$\psi^{Born}_{\rightarrow}(x) = e^{ikx} + \int_{-\infty}^{+\infty}dx'\, G_{0+}(x-x')V(x')e^{ikx'}$$

$$r^{Born}_{\rightarrow} = \frac{-ik}{2E}\langle\psi_{0\leftarrow}|\hat{U}|\psi_{0\rightarrow}\rangle = \frac{-ik}{2E}\int_{-\infty}^{+\infty}dx\, e^{2ikx}V(x)$$

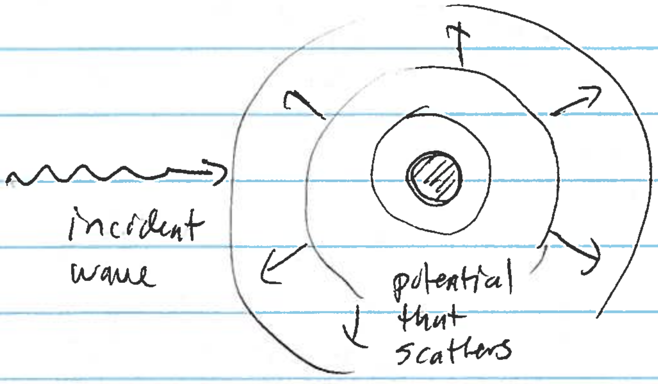
$$t^{Born}_{\rightarrow} = 1 - \frac{ik}{2E}\langle\psi_{0\rightarrow}|\hat{U}|\psi_{0\rightarrow}\rangle = 1 - \frac{ik}{2E}\int_{-\infty}^{+\infty}V(x)dx \simeq \exp\left[-\frac{ik}{2E}\int dx\, V(x)\right]$$

since we assume $\hat{V}$ is small

$$r^{Born}_{\rightarrow} \sim 0 \qquad t^{Born}_{\rightarrow} \sim 1$$

Born approximation works well when most of the wave is transmitted.

Now we discuss 3-d scattering



outgoing spherical wave like dropping a stone in a pond.

Recall expansion of plane wave in spherical harmonics

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} (2l+1)\, i^l\, j_l(kr)\, P_l(\cos\theta) \qquad E = \frac{\hbar^2 k^2}{2m} \quad k = \frac{\sqrt{2mE}}{\hbar}$$

$j_l(kr) = \sqrt{\frac{\pi}{2kr}}\, J_{l+\frac{1}{2}}(kr)$ = spherical Bessel function

$P_l(\cos\theta)$ = legondre polynomial

recall as well $j_l(kr) \rightarrow \frac{1}{kr}\sin(kr - \frac{l\pi}{2})$ for $r \rightarrow \infty$.

so $f_l(r) \rightarrow c_l \sin(kr - \frac{l\pi}{2})$ for $r \rightarrow \infty$

$$= c_l\Big(\underset{\substack{\uparrow \\ \text{outgoing wave}}}{e^{i(kr-\frac{l\pi}{2})}} - \underset{\substack{\uparrow \\ \text{incoming wave}}}{e^{-i(kr-\frac{l\pi}{2})}}\Big)$$

For interacting case need $V(r) \to 0$ faster than centrifugal potential $\frac{\hbar^2 l(l+1)}{2mr^2}$ then expect

$$f_l(r) \to \underset{\overset{\uparrow}{\text{constant}}}{A_l(k)} \left(\underset{\overset{\uparrow}{\text{incident}}}{e^{-ikr}} + \underset{\overset{\uparrow}{\text{reflected}}}{r_l(k) e^{ikr}} \right) \quad \text{as } r \to \infty$$

because nothing can transmit through $r=0$
(recall analogy to 1d with infinite wall at $r=0$)

But in 1D $R+T=1$ if $T=0 \Rightarrow R=1$

so $r = e^{i\phi}$ = phase

write $r_l(k) = -e^{i(2\delta_l(k) - l\pi)}$ $\delta_l(k)$ = lth partial wave phase shift

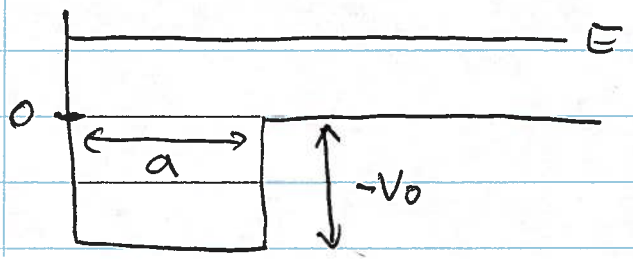
plug into find $f_l(r) \to A_l(k)\left(e^{-ikr} - e^{i(2\delta_l(k) - l\pi + kr)}\right)$

$$= A_l(k)\, e^{i(\delta_l(k) - \frac{l\pi}{2})}\, 2i \sin\left(kr + \delta_l(k) - \frac{l\pi}{2}\right)$$

$$= B_l(k) \sin\left(kr + \delta_l(k) - \frac{l\pi}{2}\right)$$

For free case, $V=0$ and $\delta_l(k)=0$ for all l and k

example: spherical well



We have $k_1 = \frac{1}{\hbar}\sqrt{2m(E+V_0)}$ for $r<a$

$k_2 = \frac{1}{\hbar}\sqrt{2mE}$ for $r>a$

$$R_\ell(r) = \begin{cases} A_\ell(E)\, j_\ell(k_1 r) & r<a \\ B_\ell(E)\, \cancel{B_\ell}\, j_\ell(k_2 r) + C_\ell(E)\, n_\ell(k_2 r) & r>a \end{cases}$$

(since no restriction on wave functions for $r>a$ since we never hit $r=0$ in this region)

continuity of ψ at $r=a$

$$A_\ell(E)\, j_\ell(k_1 a) = B_\ell(E)\, j_\ell(k_2 a) + C_\ell(E)\, n_\ell(k_2 a)$$

continuity of ψ' at $r=a$

$$k_1 A_\ell(E)\, j_\ell'(k_1 a) = k_2 B_\ell(E)\, j_\ell'(k_2 a) + k_2 C_\ell(E)\, n_\ell'(k_2 a)$$

straight forward to solve now for $\frac{B_\ell(E)}{A_\ell(E)}$ $\frac{C_\ell(E)}{A_\ell(E)}$

find

$$\frac{B_\ell(E)}{A_\ell(E)} = \frac{k_2 n_\ell'(k_2 a)\, j_\ell(k_1 a) - k_1 j_\ell'(k_1 a)\, n_\ell(k_2 a)}{k_2 n_\ell'(k_2 a)\, j_\ell(k_2 a) - k_2 j_\ell'(k_2 a)\, n_\ell(k_2 a)}$$

$$\frac{C_\ell(E)}{A_\ell(E)} = \frac{k_2 j_\ell'(k_2 a)\, j_\ell(k_1 a) - k_1 j_\ell'(k_1 a)\, j_\ell(k_2 a)}{k_2 j_\ell'(k_2 a)\, n_\ell(k_2 a) - k_2 n_\ell'(k_2 a)\, j_\ell(k_2 a)}$$

note that

$$n_\ell(\rho) \to -\frac{1}{\rho}\cos\left(\rho - \frac{\ell\pi}{2}\right)$$