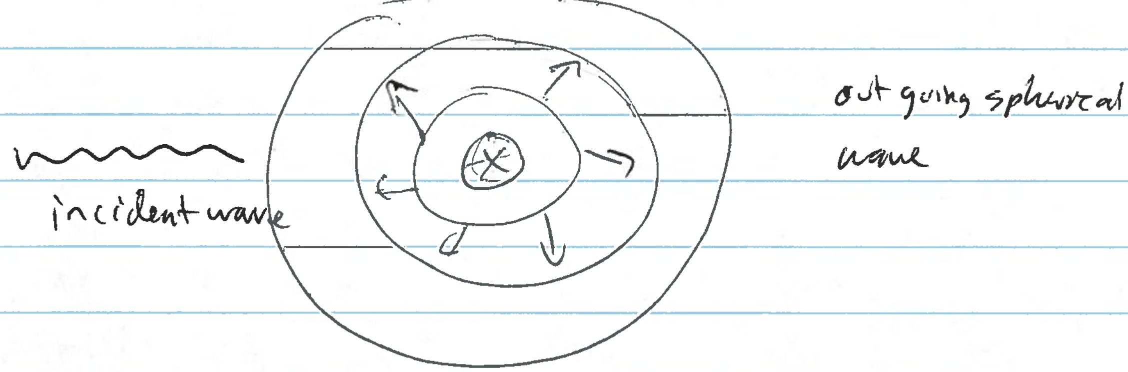
3-D Scattering.



Lippman-Schwinger equation holds

$$|\psi_k\rangle = |\psi_{0k}\rangle + \hat{G}_+(E)\,\hat{V}\,|\psi_k\rangle \qquad E = \frac{\hbar^2 k^2}{2m}$$

$$\hat{G}_{0+}(E) = \frac{1}{E - \hat{H}_0 + i\delta}$$

If we evaluate in coordinate rep in 3d we find

$$\langle \vec{r}\,|\hat{G}_{0+}(E)|\,\vec{r}\,'\rangle = -\frac{m}{2\pi\hbar^2}\,\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

the derivation requires contour integrals and will be skipped.

So $$\boxed{\psi_k(r) = \psi_{0k}(r) - \frac{1}{4\pi}\int d^3r'\,\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\,\frac{2m}{\hbar^2}V(\vec{r}\,')\,\psi_k(\vec{r}\,')}$$

for a 3d scattering case.

Now focus on the behavior for large r. If $V(r')$ is nonzero only for small $|\vec{r}'|$ and decays fast for large $|\vec{r}'|$, we can expand

$$|\vec{r}-\vec{r}'| \cong r\left|\cancel{\vec{r}} - \frac{\vec{r}'}{r}\right| = r\sqrt{1 - 2\frac{\vec{r}\cdot\vec{r}'}{r^2} + \frac{r'^2}{r^2}}$$

$$= r\left(1 - \frac{\vec{r}\cdot\vec{r}'}{r^2} - \frac{1}{2}\frac{(\vec{r}\cdot\vec{r}')^2}{r^4} + \frac{1}{2}\frac{r'^2}{r^2} + \cdots\right)$$

$$\cong r - \vec{e}_r\cdot\vec{r}' \qquad \vec{e}_r = \frac{\vec{r}}{r} = \text{unit vector in } r \text{ direction}$$

So $$\frac{1}{|\vec{r}-\vec{r}'|} \cong \frac{1}{r}\,\frac{1}{\left(1 - \vec{e}_r\cdot\vec{e}_{r'}\frac{r'}{r}\right)}$$

$$\cong \frac{1}{r}\left(1 + \vec{e}_r\cdot\vec{e}_{r'}\frac{r'}{r}\right) = \frac{1}{r} + \frac{\vec{e}_r\cdot\vec{e}_{r'}\,r'}{r^2}$$

$$k|\vec{r}-\vec{r}'| \cong kr\left(1 - \vec{e}_r\cdot\vec{e}_{r'}\frac{r'}{r} + \cdots\right)$$

define $\vec{k}' = \vec{e}_r\, k \qquad |\vec{k}'| = |\vec{k}| = k$

So $k|\vec{r}-\vec{r}'| = kr - \vec{k}'\cdot\vec{r}' + \cdots$

and

$$\psi_{\vec{k}}(\vec{r}) = \psi_{0k}(r) - \frac{1}{4\pi}\int d^3r'\,\frac{e^{ikr - i\vec{k}'\cdot\vec{r}'}}{r}\left(1 + \vec{e}_r\cdot\vec{e}_{r'}\frac{r'}{r}\right)\frac{2M}{\hbar^2}V(\vec{r}') * \psi_{k}(\vec{r}')$$

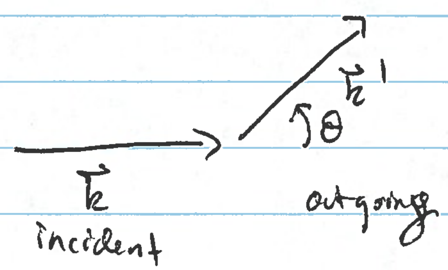
$\psi_{0\vec{k}}(\vec{r}) = \frac{1}{(2\pi)^{3/2}}e^{i\vec{k}\cdot\vec{r}}$ so

$$\psi_{\vec{k}}(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\left[e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\vec{k}',\vec{k})\right] + O\left(\left(\frac{r'}{r}\right)\right)$$

$$f(\vec{k}',\vec{k}) = -2\pi^2 \int d^3r' \, \psi^*_{0\vec{k}'}(\vec{r}') \, \frac{2m}{\hbar^2} V(\vec{r}') \, \psi_{\vec{k}}(\vec{r}')$$

$$f(\vec{k}',\vec{k}) = -\frac{4\pi^2 m}{\hbar^2} \langle \psi_{0\vec{k}'} | \hat{V} | \psi_{k} \rangle$$

$f(\vec{k}',\vec{k})$ is the scattering amplitude and it has units of length



$f(\vec{k}',\vec{k})$ is the amplitude of an outgoing spherical wave in direction $\vec{k}'$.

$|f(\vec{k}',\vec{k})|^2$ = probability to observe particle with momentum $\hbar\vec{k}'$ after scattering (incident momentum is $\hbar\vec{k}$).

The differential cross section is defined via

$$\frac{d\sigma}{d\Omega_{\vec{k}'}} = \frac{\text{prob/time/solid angle of scattering in } \vec{k}' \text{ direction}}{\text{prob/time/area of incident flux of particles}}$$

σ = cross section = $\int d\Omega \frac{d\sigma}{d\Omega}$ has units of area.

The cross section is a function of the incident momentum $\hbar\vec{k}$ or incident energy E, but doesn't depend on the frame of reference.

Gottfried discusses how we find

$$\frac{d\sigma}{d\Omega_{\vec{k}'}} = |f(\vec{k}',\vec{k})|^2$$

These results are often summarized in terms of the transition matrix or T matrix.

$$|\psi_k\rangle = |\psi_{0k}\rangle + \hat{G}_{0+}(E)\hat{V}|\psi_{\vec{k}}\rangle$$

$$\Rightarrow |\psi_k\rangle = [1 - \hat{G}_{0+}(E)\hat{V}]^{-1}|\psi_{0k}\rangle = \hat{\Omega}_+(E)|\psi_{0k}\rangle$$

$\hat{\Omega}_+(E)$ is called the Möller wave matrix

$$f(\vec{k}',\vec{k}) = \frac{-4\pi^2 m}{\hbar^2}\langle \phi_{0\vec{k}'}|\hat{V}|\psi_k\rangle = \frac{-4\pi^2 m}{\hbar^2}\langle \psi_{0\vec{k}'}|\hat{V}\hat{\Omega}_+(E)|\psi_{0\vec{k}}\rangle$$

$$\hat{T}(E) = \hat{V}\hat{\Omega}_+(E) = \text{T matrix}$$

$$f(\vec{k}',\vec{k}) = \frac{-4\pi^2 m}{\hbar^2}\langle \psi_{0\vec{k}'}|\hat{T}(E)|\psi_{0\vec{k}}\rangle = -\frac{4\pi^2 m}{\hbar^2} T_{\vec{k}'\vec{k}}(E)$$

$\hat{T}(E)$ satisfies operator equations:

$$\hat{T} = \hat{V}[1 - \hat{G}_{0+}\hat{V}]^{-1}$$

$$\Rightarrow \hat{T} - \hat{T}\hat{G}_{0+}\hat{V} = \hat{V} \quad \text{or} \quad \boxed{\hat{T} = \hat{V} + \hat{T}\hat{G}_{0+}\hat{V}}$$

also $\hat{T} = \hat{V} + \hat{V}\hat{G}_{0+}\hat{V} + \hat{V}\hat{G}_{0+}\hat{V}\hat{G}_{0+}\hat{V} + \cdots$

$$= [1 - \hat{V}\hat{G}_{0+}]^{-1}\hat{V}$$

so $[1 - \hat{V}\hat{G}_{0+}]\hat{T} = \hat{V}$ or $\boxed{\hat{T} = \hat{V} + \hat{V}\hat{G}_{0+}\hat{T}}$

since $\hat{G}_{0\pm}(E) = \dfrac{1}{E - \hat{H}_0 \pm i\delta}$, we have

$$\hat{G}_{0+}^{\dagger}(E) = \hat{G}_{0-}(E)$$

Then taking Hermitian conjugates yields

$$\hat{T}^{\dagger} = \hat{V} + \hat{T}^{\dagger}\hat{G}_{0-}\hat{V} = \hat{V} + \hat{V}\hat{G}_{0-}\hat{T}^{\dagger}$$

recall $\dfrac{1}{x \pm i\delta} = \dfrac{P}{x} \mp i\pi\delta(x)$ dirac identity

principal value

so

$$\hat{T} - \hat{T}^{\dagger} = \cancel{\hat{V}} + \hat{V}\hat{G}_{0+}\hat{T} - \cancel{\hat{V}} - \hat{T}^{\dagger}\hat{G}_{0-}\hat{V}$$

$$= \hat{T}^{\dagger}\hat{G}_{0+}\hat{T} - \cancel{\hat{T}^{\dagger}\hat{G}_{0-}\hat{V}\hat{G}_{0+}\hat{T}} - \hat{T}^{\dagger}\hat{G}_{0-}\hat{T} + \cancel{\hat{T}^{\dagger}\hat{G}_{0-}\hat{V}\hat{G}_{0+}\hat{T}}$$

$$= \hat{T}^{\dagger}\left(\hat{G}_{0+} - \hat{G}_{0-}\right)\hat{T}$$

But $\hat{G}_{0+} - \hat{G}_{0-} = \dfrac{P}{E-\hat{H}_0} - i\pi\delta(E-\hat{H}_0) - \dfrac{P}{E-\hat{H}_0} - i\pi\delta(E-\hat{H}_0)$

$$= -2i\pi\,\delta(E-\hat{H}_0)$$

$$\boxed{\hat{T} - \hat{T}^{\dagger} = -2\pi i\,\hat{T}^{\dagger}\,\delta(E-\hat{H}_0)\,\hat{T}}$$

called the generalized optical theorem.

Take matrix elements with $\langle \psi_{0\vec{k}'}|$ $|\psi_{0\vec{k}}\rangle$

and introduce complete sets of states on both sides of the delta function.

$$T_{\vec{k}'\vec{k}}(E) - T^*_{\vec{k},\vec{k}'}(E) = -2i\pi \int d^3k'' \int d^3k''' \; T^*_{\vec{k}'',\vec{k}'}(E) \; \langle \vec{k}''|\delta(E-\hat{H}_0)|\vec{k}'''\rangle T_{\vec{k}''',\vec{k}}(E)$$

but $\langle \vec{k}''|\delta(E-\hat{H}_0)|\vec{k}'''\rangle = \delta(E - \frac{\hbar^2 \vec{k}''^2}{2m}) \langle \vec{k}''|\vec{k}'''\rangle = \delta(E - \frac{\hbar^2 \vec{k}''^2}{2m}) \delta^3(\vec{k}''-\vec{k}''')$

so the integral can be done over $\vec{k}'''$

Furthermore, $\int d^3\vec{k}'' \; \delta(E - \frac{\hbar^2 \vec{k}''^2}{2m}) = \int_0^\infty dk'' \, k''^2 \int d\Omega_{\vec{k}''} \, \delta(E - \frac{\hbar^2 k''^2}{2m})$

$$= \frac{2m}{\hbar^2} \frac{\sqrt{2mE}}{\hbar} \int d\Omega_{\vec{k}''} = \frac{2mk}{\hbar^2} \int d\Omega_{\vec{k}''}$$

so we get

$$T_{\vec{k}'\vec{k}}(E) - T^*_{\vec{k}\vec{k}'}(E) = -\frac{2i\pi mk}{\hbar^2} \int d\Omega_{\vec{k}''} \; T^*_{\vec{k}''\vec{k}'}(E) \; T_{\vec{k}''\vec{k}}(E)$$

Multiply by $-\frac{4\pi^2 m}{\hbar^2}$ to get

$$\boxed{f(\vec{k}',\vec{k}) - f^*(\vec{k},\vec{k}') = \frac{ik}{2\pi} \int d\Omega_{\vec{k}''} \; f(\vec{k}'',\vec{k}) \, f^*(\vec{k}'',\vec{k}')}$$

generalized optical theorem for scattering amplitudes.

Forward scattering — $\vec{k} = \vec{k}'$, $\theta = 0$

$$2 \, \mathrm{Im} \, f(\vec{k},\vec{k}) = \frac{k}{2\pi} \int d\Omega_{\vec{k}''} \, |f(\vec{k}'',\vec{k})|^2$$

$$= \frac{k}{2\pi} \int d\Omega_{\vec{k}''} \frac{d\sigma(k)}{d\Omega_{\vec{k}''}} = \frac{k}{2\pi} \sigma(k)$$

so $\boxed{\sigma(k) = \frac{4\pi}{k} \, \mathrm{Im} \, f(\vec{k},\vec{k})}$

Born Series

$$\hat{T} = (1-\hat{V}\hat{G}_{0+})^{-1}\hat{V}$$

$$f(\vec{k}',\vec{k}) = -\frac{4\pi^2 m}{\hbar^2}\langle \psi_{0\vec{k}'}|(1-\hat{V}\hat{G}_{0+}(E))^{-1}\hat{V}|\psi_{0\vec{k}}\rangle$$

$$= \sum_{n=1}^{\infty} f_n(\vec{k}',\vec{k}) \qquad n \text{ counts powers of } \hat{V}$$

$$f^{(N)}(\vec{k}',\vec{k}) = N\text{th Born approximation}$$

$$= \sum_{n=1}^{N} f_n(\vec{k}',\vec{k})$$

$$f^{(1)}(\vec{k}',\vec{k}) = -\frac{4\pi^2 m}{\hbar^2}\langle \psi_{0\vec{k}'}|\hat{V}|\psi_{0\vec{k}}\rangle$$

$$= -\frac{4\pi^2 m}{\hbar^2}\cdot\frac{1}{(2\pi)^3}\int d^3r\, e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}}\, V(\vec{r})$$

$$= -\frac{m}{2\pi\hbar^2}\int d^3r\, e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}}\, V(\vec{r}) = \tilde{V}(\vec{k}'-\vec{k})$$

Fourier transform

Note $\langle \psi_{0\vec{k}'} | \hat{V} | \psi_{0\vec{k}} \rangle = \dfrac{\tilde{V}(\vec{k}'-\vec{k})}{(2\pi)^3}$ so

$$f^{(2)}(\vec{k}',\vec{k}) = -\frac{m}{2\pi\hbar^2}\Big[\tilde{V}(\vec{k}'-\vec{k}) + (2\pi)^3\int d^3k'' \langle \psi_{0\vec{k}'} | \hat{V} | \psi_{0\vec{k}''} \rangle \langle \psi_{0\vec{k}''} | \hat{G}_{0+}(E)\hat{V} | \psi_{0\vec{k}} \rangle \Big]$$

but $\langle \psi_{0\vec{k}''} | \hat{G}_{0+}(E) = \dfrac{1}{E - \frac{\hbar^2 k''^2}{2m} + i\delta} \langle \psi_{0\vec{k}''} |$ so

$$f^{(2)}(\vec{k}',\vec{k}) = -\frac{m}{2\pi\hbar^2}\Big[\tilde{V}(\vec{k}'-\vec{k}) + \int \frac{d^3k''}{(2\pi)^3} \frac{\tilde{V}(\vec{k}'-\vec{k}'')\,\tilde{V}(\vec{k}''-\vec{k})}{E - \frac{\hbar^2 k''^2}{2m} + i\delta}\Big]$$

and so on.

called off shell scattering because some of the integrand contributes when $\frac{\hbar^2 k''^2}{2m} \neq E$