Physics 506 lecture 20

Scattering resonances

Last time we showed from the delta shell solution that at low energy

$$\tan \delta_\ell(k) \sim (ka)^{2\ell+1}$$

This is called the Wigner threshold law and it shows that s-wave scattering dominates at low energy.

We parametrize the low energy scattering with a scattering length and an effective range via

$$k^{2\ell+1} \cot \delta_\ell(k) = -\frac{1}{a_\ell} + \frac{1}{2} k r_\ell^2 + \text{higher order in } k$$

a_ℓ and r_ℓ are constants.

a_0 = scattering length
r_0 = effective range

recall we found

$$\tan \delta_\ell(k) = \frac{\lambda a (ka)^{2\ell+1}}{[(2\ell+1)!!]^2} \frac{1}{1 - \frac{\lambda a}{2\ell+1}}$$

for the delta shell potential.

Hence

$$k^{2\ell+1} \cot \delta_\ell(k) = \frac{[(2\ell+1)!!]^2}{\lambda a} \frac{\left(1 - \frac{\lambda a}{2\ell+1}\right)}{a^{2\ell+1}}$$

$$\Rightarrow \quad a_\ell = -\frac{\lambda a}{1 - \frac{\lambda a}{2\ell+1}} \frac{a^{2\ell+1}}{[(2\ell+1)!!]^2}$$

$$a_0 = -\frac{\lambda a}{1 - \lambda a} \qquad a_0 = \boxed{\frac{-\lambda a^2}{1 - \lambda a} = a_0}$$

Note that the scattering length goes through a divergence when $\lambda a = 1$

The higher partial waves diverge when $\lambda a = 2\ell+1$

Hence the scattering length can be much larger than the range of the potential.

These resonances, called Feshbach resonances, when tuned by an external magnetic field are very important in ultra-cold atomic physics as they can govern the interactions between atoms and allow one to tune different interactions.

As $k \to 0$ $\quad \sigma(k) \to 4\pi \dfrac{\sin^2\delta_0(k)}{k^2}$

But $\delta_0(k) \sim \dfrac{\lambda a\, ka}{1-\lambda a} = -a_0 k$

so $\quad \sigma(k\to 0) = 4\pi a_0^2$

So the scattering length determines the total cross section for low energy scattering.

What is the origin of the scattering resonance?

It is related to the bound states of the potential.

Examine the bound-state delta shell potential

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} f_{n\ell}(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} f_{n\ell}(r) - \frac{\hbar^2\lambda}{2m}\delta(r-a) f_{n\ell}(r) = E_{n\ell} f_{n\ell}(r)$$

We already saw for the plane wave case that the solutions were $j_\ell(ikr)$ $\eta_\ell(ikr)$ when $r \neq a$ $\quad \left(-\frac{\hbar^2 k^2}{2m} = E\right)$ analytically continue to imag argument

We matched the wave functions at $r=a$ and took into account the discontinuity at $r=a$

pick $\quad f_{n\ell}(r) = A\, j_\ell(ikr) \qquad r<a$

$\qquad = B\, j_\ell(ikr) + C\, \eta_\ell(ikr) \qquad r>a$

~~$A j_\ell(ika) = B j_\ell(ika) + C \eta_\ell(ika)$~~

the behavior as $r \to 0$ is fine since
$$j_\ell \sim (kr)^\ell$$

for large r, $j_\ell(ikr) \sim \frac{1}{ikr}\sin\left(ikr - \frac{\ell\pi}{2}\right)$
$$\eta_\ell(ikr) \sim -\frac{1}{ikr}\cos\left(ikr - \frac{\ell\pi}{2}\right)$$

$\Rightarrow$ need $C = iB$ so it decays exponentially

~~$i\cos x + \sin x \sim$~~ $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ $\quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$

$\Rightarrow \frac{1}{i}\cos x + \sin x \sim e^{+ix}$ $\quad x = ikr \Rightarrow$ exponential decay

so
$$f_{n\ell}(r) = \begin{cases} A j_\ell(ikr) & r<a \\ B\left(j_\ell(ikr) + i\eta_\ell(ikr)\right) & r>a \end{cases}$$

continuity at $r=a \Rightarrow A j_\ell(ika) = B\left(j_\ell(ika) + i\eta_\ell(ika)\right)$
derivative disc at $r=a \Rightarrow Bik\left[j_\ell'(ika) + i\eta_\ell'(ika)\right] - Aik\, j_\ell'(ika)$
$$= -\lambda A j_\ell(ika)$$

$$A = B\left(1 + i\frac{\eta_\ell(ika)}{j_\ell(ika)}\right)$$

$$B\left[j_\ell'(ika) + i\eta_\ell'(ika)\right] - B\left(1 + i\frac{\eta_\ell(ika)}{j_\ell(ika)}\right)j_\ell'(ika)$$
$$= i\frac{\lambda}{k}B\left(1 + i\frac{\eta_\ell(ika)}{j_\ell(ika)}\right)j_\ell(ika)$$

$$i j_\ell(ika)\eta_\ell'(ika) - i\eta_\ell(ika)j_\ell'(ika) = i\frac{\lambda}{k}j_\ell(ika)\left(j_\ell(ika) + i\eta_\ell(ika)\right)$$

examine as $k \to 0$

$$j_\ell(ika) \to \frac{(ika)^\ell}{(2\ell+1)!!} \qquad j_\ell'(ika) \to \frac{\ell}{ika} j_\ell(ika)$$

$$\eta_\ell(ika) \to -\frac{(2\ell+1)!!}{(ika)^{\ell+1}} \qquad \eta_\ell'(ika) \to -\frac{(\ell+1)}{ika}\eta_\ell(ika)$$

$$j_\ell(ika)\,\eta_\ell(ika) \to -\frac{1}{ika}\,\frac{1}{2\ell+1}$$

so we get

$$-\frac{(\ell+1)}{ika}\left(-\frac{1}{ika}\right)\frac{1}{2\ell+1} - \frac{\ell}{ika}\left(\frac{-1}{ika}\right)\frac{1}{2\ell+1} = \frac{\lambda}{k} i \left(\frac{-1}{ika}\right)\frac{1}{2\ell+1}$$

$$(2\ell+1) = \lambda a$$

so when $\lambda a = 2\ell+1$ a bound state appears at $E=0$!

So there is a connection between the appearance of a bound state and the divergence of a scattering amplitude as $k \to 0$

In fact, if we think of the scattering amplitudes as functions of complex k, then bound states appear at the poles of the scattering amplitudes, which is an alternate way to solve bound state problems!

For the delta shell potential, we have

$$f_\ell(k) = \frac{1}{k} \frac{ka\, \lambda a\, j_\ell^2(ka)}{1 - i\, ka\, \lambda a\, j_\ell(ka)\left(j_\ell(ka) + i\eta_\ell(ka)\right)}$$

$\Rightarrow$ poles at $\quad 1 = i\, ka\, \lambda a\, j_\ell(ka)\left(j_\ell(ka) + i\eta_\ell(ka)\right)$

need to substitute $k \to iK$ and solve for poles,

This will give the formula for the bound state energies

$$1 = -Ka\, \lambda a\, j_\ell(iKa)\left(j_\ell(iKa) + i\eta_\ell(iKa)\right)$$

The general rule is a positive scattering length $\Rightarrow$ weakly bound state and strong scattering near $k=0$

a negative scattering length $\Rightarrow$ a pre bound state and strong scattering near $k=0$

In atomic physics, one sweeps the magnetic field across the resonance and forms bound state molecules (weakly bound) which can then be studied or made into more deeply bound objects.

We can show this condition hold by verifying the wronskian

$$j_\ell n_\ell' - n_\ell j_\ell' = \frac{1}{z^2}$$

Proof:

$$z^2 \frac{d^2 n_\ell}{dz^2} + 2z \frac{d n_\ell}{dz} + (z^2 - \ell(\ell+1)) n_\ell = 0 \quad \text{multiply by } j_\ell$$

$$z^2 \frac{d^2 j_\ell}{dz^2} + 2z \frac{d j_\ell}{dz} + (z^2 - \ell(\ell+1)) j_\ell = 0 \quad \text{multiply by } n_\ell$$

Subtract to get

$$z^2 \left[j_\ell n_\ell'' - n_\ell j_\ell'' \right] + 2z \left[j_\ell n_\ell' - n_\ell j_\ell' \right] = 0$$

$$z^2 \frac{d}{dz} \left[j_\ell n_\ell' - n_\ell j_\ell' \right] + 2z \left[j_\ell n_\ell' - n_\ell j_\ell' \right] = 0$$

$$-\frac{2}{z} dz = d \ln \left[j_\ell n_\ell' - n_\ell j_\ell' \right]$$

$$-2 \ln z + c = \ln \left[j_\ell n_\ell' - n_\ell j_\ell' \right]$$

$$\Rightarrow \quad \frac{c}{z^2} = j_\ell n_\ell' - n_\ell j_\ell'$$

checking the limit $z \to 0$ gives $c = 1$

so $\quad j_\ell n_\ell' - n_\ell j_\ell' = \frac{1}{z^2} \quad$ or

$$-\frac{1}{(ka)^2} = \frac{\lambda}{k} j_\ell(ika) \left[j_\ell(ika) + i n_\ell(ika) \right]$$

$$1 = -\lambda a ka \, j_\ell(ika) \left[j_\ell(ika) + i n_\ell(ika) \right]$$

take $ka \to ika$ in scattering amplitude to find same equation