Interaction representation

Last time we developed an expansion for $U(t,t_0)$ in powers of $\hat{H}(t)$

$$|\psi(t)\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle$$

$$\hat{U}(t,t_0) = T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\,\hat{H}(t')\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n \frac{1}{n!}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n\, T\left(\hat{H}(t_1)\cdots\hat{H}(t_n)\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, \hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n)$$

Note also that the equation of motion for U is

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = \hat{H}(t)\,U(t,t_0)$$

which follows from the last form for U

t only appears here

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n i\hbar\frac{\partial}{\partial t}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, \hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n)$$

$$= \frac{-i}{\hbar}\, i\hbar\, \hat{H}(t) \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_3 \cdots \int_{t_0}^{t_{n-1}} dt_n\, \hat{H}(t_2)\cdots\hat{H}(t_n)$$

$$= \hat{H}(t)\,U(t,t_0)$$

But in many case $\hat{H} = \hat{H}_0 + \hat{V}(t)$ ← possibly small

$\hat{H}_0$: time independent

In those cases we want an expansion in $\hat{V}$ not $\hat{H}$.

We start by looking at different pictures for quantum mechanics.

We are all familiar with the Schroedinger rep.

$$\hat{H}|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle$$ assume $\hat{H}$ independent of time here

Now consider the expectation value of an operator $\hat{A}$ with no time dependence.

$$A(t) = \langle\psi(t)|\hat{A}|\psi(t)\rangle$$

$\uparrow$ function of t $\quad\quad$ $\uparrow$ operator

$$\frac{d}{dt}A(t) = \left(\frac{\partial}{\partial t}\langle\psi(t)|\right)\hat{A}|\psi(t)\rangle + \langle\psi(t)|\hat{A}\frac{\partial}{\partial t}|\psi(t)\rangle$$

$$= \frac{i}{\hbar}\langle\psi(t)|\hat{H}\hat{A}|\psi(t)\rangle - \frac{i}{\hbar}\langle\psi(t)|\hat{A}\hat{H}|\psi(t)\rangle$$

$$= -\frac{i}{\hbar}\langle\psi(t)[\hat{A},\hat{H}]|\psi(t)\rangle$$

$$i\hbar\frac{d}{dt}A(t) = \langle\psi(t)[\hat{A},\hat{H}|\psi(t)\rangle$$

all time dependence comes from the wave functions in the schroedinger representation when $\frac{\partial H}{\partial t} = 0$.

Heisenberg representation

write $\quad |\psi_s(t)\rangle = \hat{U}(t)\,|\psi_H\rangle \qquad \hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$

Then define

$$\hat{A}_H(t) = \hat{U}^+(t)\,\hat{A}\,U(t)$$

Then $A(t) = \langle\psi_s(t)|\,\hat{A}\,|\psi_s(t)\rangle$

$$= \langle\psi_H|\,U^+(t)\,\hat{A}\,U(t)\,|\psi_H\rangle$$

$$= \langle\psi_H|\,\hat{A}_H(t)\,|\psi_H\rangle$$

and all time dependence is in the operators now.

$$i\hbar\frac{d}{dt}\hat{A}_H(t) = i\hbar\frac{\partial}{\partial t}\hat{U}^+\,\hat{A}\,\hat{U} + \hat{U}^+\hat{A}\;i\hbar\frac{\partial}{\partial t}\hat{U}$$

$$= -\hat{U}^+\hat{H}\hat{A}\hat{U} + \hat{U}^+\hat{A}\hat{H}\hat{U}$$

$$= \hat{U}^+[\hat{A},\hat{H}]\,U$$

$$= [\hat{A}_H(t),\hat{H}] \quad \text{since } [\hat{U},H] = 0 \text{ for time independent } \hat{H}.$$

$$\boxed{i\hbar\frac{d}{dt}\hat{A}_H(t) = [\hat{A}_H(t),\hat{H}]}$$

All time dependence is in the operators which evolve like Poisson brackets in classical mechanics.

If $\hat{H}$ depends on time $\hat{H}_S(t)$ we proceed similarly

$$|\psi_S(t)\rangle = \hat{U}(t,t_0)\,|\psi_H(t_0)\rangle$$
$$\hat{A}_H(t) = \hat{U}^+(t,t_0)\,\hat{A}_S(t)\,\hat{U}(t,t_0)$$
$$\hat{H}_H(t) = \hat{U}^+(t,t_0)\,\hat{H}_S(t)\,\hat{U}(t,t_0)$$

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}_S(t)\,\hat{U}(t,t_0)$$

so $i\hbar\frac{\partial}{\partial t}\hat{A}_H(t) = i\hbar\frac{\partial}{\partial t}\hat{U}^+(t,t_0)\,\hat{A}_S(t)\,U(t,t_0)$

$$+ i\hbar\,\hat{U}^+(t,t_0)\frac{\partial}{\partial t}\hat{A}_S(t)\,\hat{U}(t,t_0) + i\hbar\,\hat{U}^+(t,t_0)\,\hat{A}_S(t)\frac{\partial}{\partial t}\hat{U}(t,t_0)$$
$$= -\hat{U}^+(t,t_0)\,\hat{H}_S(t)\,\hat{A}_S(t)\,\hat{U}(t,t_0)$$
$$+ i\hbar\,\hat{U}^+(t,t_0)\frac{\partial}{\partial t}\hat{A}_S(t)\,\hat{U}(t,t_0) + \hat{U}^+(t,t_0)\hat{A}_S(t)\,\hat{H}_S(t)\,\hat{U}(t,t_0)$$

but $\hat{U}^+\hat{H}_S\hat{A}_S\hat{U} = \hat{U}^+\hat{H}_S\hat{U}^+\hat{U}^+\hat{A}_S\hat{U} = \hat{H}_H(t)\,\hat{A}_H(t)$ so

$$\boxed{i\hbar\frac{\partial}{\partial t}\hat{A}_H(t) = [\hat{A}_H(t), \hat{H}_H(t)] + i\hbar\frac{\partial\hat{A}_H}{\partial t}(t)}$$

The last term is $i\hbar\,\hat{U}^+(t,t_0)\frac{\partial A_S(t)}{\partial t}\hat{U}(t,t_0) = i\hbar\frac{\partial\hat{A}_H(t)}{\partial t}$

Interaction representation

The interaction representation is halfway between Schroedinger and Heisenberg.

$$\hat{H}(t) = \underset{\uparrow}{\hat{H}_0} + \underset{\uparrow}{\hat{V}(t)}$$

No time dep   time dependent but small

So we have

$$\hat{H}_0 |n\rangle = E_n^0 |n\rangle$$

Define $\quad |\psi_I(t)\rangle = e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)} |\psi_S(t)\rangle$

$$\hat{A}_I(t) = e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)} A_S(t)\, e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}$$

$$\hat{V}_I(t) = e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)} \hat{V}_S(t)\, e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}$$

Note that $\quad H_{0I}(t) = H_{0S}(t) = H_0(t) \quad$ since

$$[\hat{H}_0, e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}] = 0.$$

$$i\hbar \frac{\partial}{\partial t} |\psi_I(t)\rangle = i\hbar \frac{d}{dt}\left[e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)} |\psi_S(t)\rangle \right]$$

$$= -\hat{H}_0 e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)} |\psi_S(t)\rangle + e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)} \hat{H}_S(t) |\psi_S(t)\rangle$$

$$= e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)} \left(\cancel{-\hat{H}_0} + \cancel{\hat{H}_0} + \hat{V}_S(t) \right) |\psi_S(t)\rangle$$

$$= e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)} \hat{V}_S(t)\, e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)} e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)} |\psi_S(t)\rangle$$

$$= \hat{V}_I(t) |\psi_I(t)\rangle$$

So, If $\quad \hat{U}_I(t,t_0)|\psi_I(t_0)\rangle = |\psi_I(t)\rangle$, then we have

$$i\hbar \frac{\partial}{\partial t} \hat{U}_I(t,t_0) = \hat{V}_I(t)\, \hat{U}_I(t,t_0)$$

$$\Rightarrow \boxed{\begin{aligned} &\hat{U}_I(t,t_0) = T \exp\left[-\frac{i}{\hbar} \int_{t_0}^{t} \hat{V}_I(t')\,dt' \right] \\ &\hat{U}_I(t,t_0) = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \\ &\qquad * T\left[\hat{U}_I(t_1)\, \hat{V}_I(t_2) \cdots \hat{V}_I(t_n) \right] \end{aligned}}$$

This is called the Dyson expansion for the evolution operator.

Note that

$$|\psi_I(t)\rangle = e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}|\psi_S(t)\rangle = e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\hat{U}_S(t,t_0)|\psi_S(t_0)\rangle$$

$$= \hat{U}_I(t,t_0)\,|\psi_I(t_0)\rangle$$

but $|\psi_I(t_0)\rangle = |\psi_S(t_0)\rangle = |\psi_H(t_0)\rangle$ since $U(t_0,t_0)=1$ in all pictures.

so $$\boxed{U_S(t,t_0) = e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}\,T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}dt'\,\hat{V}_I(t')\right]}$$

This is the desired expansion of the evolution operator in a power series in $\hat{V}$ and it will be used for time dependent perturbation theory in a few lectures.

We can derive this result directly though.

recall $$i\hbar\frac{\partial}{\partial t}\,T e^{-\frac{i}{\hbar}\int_{t_0}^{t}dt'\hat{A}(t')} = \hat{A}(t)\,T e^{-\frac{i}{\hbar}\int_{t_0}^{t}dt'\hat{A}(t')}$$

so

$$i\hbar\frac{d}{dt}\left[e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}\,T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}dt'\,\hat{V}_I(t')\right]\right]$$

$$= \hat{H}_0\, e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}\,T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{V}_I(t')dt'\right]$$
$$+ e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}\,\hat{V}_I(t)\,T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{V}_I(t')dt'\right]$$

but $e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}\hat{V}_I(t) = \hat{V}_s(t)\, e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}$

so

$$= \underbrace{[\hat{H}_0 + \hat{V}_s(t)]}_{\hat{H}(t)}\, e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}\, T e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt' \hat{V}_I(t')}$$

so $i\hbar \frac{\partial}{\partial t} O_p(t,t_0) = \hat{H}(t)\, O_p(t,t_0)$ and $O_p(t_0,t_0) = \mathbb{1}$

$\Rightarrow O_p(t,t_0) = \hat{U}_s(t,t_0)$

So we have proved the result directly.