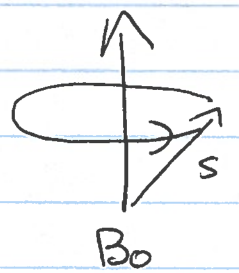
Cyclotron resonance

An example of an exact time-ordered product.

When a spin is placed in a constant magnetic field it precesses



The magnetic moment of a spin, with angular momentum $\vec{S}$ is $\vec{\mu} = \gamma \vec{S}$ (Gottfried normalization for $\vec{S}$)

γ = gyromagnetic ratio = $\dfrac{eg\hbar}{2mc}$ $\quad g$ = Landé g factor

$$\hat{H}_0 = -\vec{\mu}\cdot\vec{B}_0 = -\gamma B_0 \hat{S}_z \qquad \vec{B}_0 \text{ along } \hat{z} \text{ direction}$$

$$= -\hbar\Omega\hat{S}_z \qquad \Omega = \frac{\gamma B_0}{\hbar} = \text{Larmor frequency.}$$

If our states had total spin S and z component m then

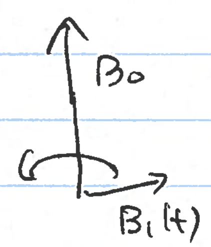
$$\hat{H}_0 |s,m\rangle = -\hbar\Omega m |s,m\rangle$$

The time dependence is

$$|s,m;t\rangle = e^{-\frac{i}{\hbar}\hat{H}_0 t}|s,m\rangle = e^{i\Omega m t}|s,m;t=0\rangle$$

States rotate with a frequency Ωm. This rotation is called precession (harmonics of the Larmor frequency).

In a magnetic resonance experiment, we add an additional small perpendicular field that rotates around the z axis



The second field is small, but its rotation frequency can be adjusted externally. Interesting things happen when the second field rotates at the same rate as the precession of the spin. Then, in the spin's rest frame, it sees a static perpendicular magnetic field, which the spin will precess about. If we wait long enough, the spin will flip.

Let's examine mathematically:

$$\vec{B}(t) = B_0 \hat{z} + B_1 (\hat{x} \cos\omega t - \hat{y} \sin\omega t)$$

$$\hat{H}(t) = -\vec{\mu}\cdot\vec{B} = -\hbar\Omega \hat{S}_z - \gamma B_1 (\hat{S}_x \cos\omega t - \hat{S}_y \sin\omega t)$$

Now consider the operator that transforms us into the rotating frame of the $\perp$ magnetic field B_1

$$\hat{D}(\omega t, \hat{z}) = e^{-i\omega t \hat{S}_z}$$

$$\hat{D}^+(\omega t, \hat{z})\, \hat{S}_z\, \hat{D}(\omega t, \hat{z}) = e^{i\omega t \hat{S}_z} \hat{S}_z e^{-i\omega t \hat{S}_z} = \hat{S}_z$$

$$\hat{D}^+(\omega t, \hat{z})\, \hat{S}_x\, \hat{D}(\omega t, \hat{z}) = e^{i\omega t \hat{S}_z} \hat{S}_x e^{-i\omega t \hat{S}_z} = f_x(t)$$

$$\frac{df_x}{dt} = i\omega\, e^{i\omega t \hat{S}_z} [\hat{S}_z, \hat{S}_x] e^{-i\omega t \hat{S}_z}$$

but $[\hat{S}_z, \hat{S}_x] = i\hat{S}_y$ Gottfried normalization

$$\frac{df_x}{dt} = -\omega f_y \qquad f_y = e^{i\omega t \hat{S}_z} \hat{S}_y e^{-i\omega t \hat{S}_z}$$

$$\frac{df_y}{dt} = i\omega\, e^{i\omega t \hat{S}_z} [\hat{S}_z, \hat{S}_y] e^{-i\omega t \hat{S}_z} = \omega f_x$$

so $$\frac{d^2 f_x}{dt^2} = -\omega^2 f_x \qquad \frac{d^2 f_y}{dt^2} = -\omega^2 f_y$$

$$f_x(t) = \cos\omega t\, f_x(0) + \frac{1}{\omega} \sin\omega t\, \dot{f}_x(0)$$

$$f_y(t) = \cos\omega t\, f_y(0) + \frac{1}{\omega} \sin\omega t\, \dot{f}_y(0)$$

$$f_x(0) = \hat{S}_x \qquad \dot{f}_x(0) = -\omega \hat{S}_y$$
$$f_y(0) = \hat{S}_y \qquad \dot{f}_y(0) = \omega \hat{S}_x$$

so $$\hat{D}^+(\omega t, \hat{z})\, \hat{S}_x\, \hat{D}(\omega t, \hat{z}) = \cos\omega t\, \hat{S}_x - \sin\omega t\, \hat{S}_y$$
$$\hat{D}^+(\omega t, \hat{z})\, \hat{S}_y\, \hat{D}(\omega t, \hat{z}) = \cos\omega t\, \hat{S}_y + \sin\omega t\, \hat{S}_x$$

Hence, we can write

$$\boxed{\hat{H}(t) = -\ \hat{D}^+(\omega t, \hat{z})\left[-\hbar\Omega \hat{S}_z - \gamma B_1 \hat{S}_x\right]\hat{D}(\omega t, \hat{z})}$$

And the Hamiltonian is a unitary transformation of a time-independent Hamiltonian!

So, we write
$$|\psi(t)\rangle = \hat{D}^+(\omega t, \hat{z})\, |\psi_R(t)\rangle \qquad R = \text{rotating}$$

$$\hat{H}(t)|\psi(t)\rangle = \hat{D}^+(\omega t, \hat{z})\left[-\hbar\Omega \hat{S}_z - \gamma B_1 \hat{S}_x\right]|\psi_R(t)\rangle \quad \text{since } \hat{D}\hat{D}^+ = 1.$$

$$= i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle$$

multiply by $\hat{D}(\omega t, \hat{z})$ to get

$$\left[-\hbar\Omega \hat{S}_z - \gamma B_1 \hat{S}_x\right]|\psi_R(t)\rangle = i\hbar\, \hat{D}(\omega t, \hat{z}) \frac{\partial}{\partial t} \hat{D}^+(\omega t, \hat{z})|\psi_R(t)\rangle$$
$$= -\hbar\omega \hat{S}_z |\psi_R(t)\rangle + i\hbar \frac{\partial}{\partial t}|\psi_R(t)\rangle$$

$$\left[(\hbar\omega - \hbar\Omega)\hat{S}_z - \gamma B_1 \hat{S}_x\right]|\psi_R(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi_R(t)\rangle$$

↑ time independent, so easy to solve

$$|\psi_R(t)\rangle = e^{-\frac{i}{\hbar}t\left[(\hbar\omega-\hbar\Omega)\hat{S}_z - \gamma B_1\hat{S}_x\right]}|\psi_R(0)\rangle$$

$$\uparrow \; e^{-\frac{i}{\hbar}\hat{H}'t}|\psi_R(0)\rangle$$

So $|\psi(t)\rangle = \hat{D}^+(\omega t, \hat{z})|\psi_R(t)\rangle$ or

$$\boxed{|\psi(t)\rangle = e^{i\omega t\hat{S}_z} e^{-it\left[(\omega-\Omega)\hat{S}_z - \frac{\gamma B_1}{\hbar}\hat{S}_x\right]}|\psi_R(0)\rangle}$$

Suppose we start with our system in a pure state $|jm\rangle$ at $t=0$. What is the probability to be in another state at time t?

$$P(t)_{j'm' \leftarrow jm} = \left|\langle j'm'|\, e^{i\omega t\hat{S}_z} e^{-it\left[(\omega-\Omega)\hat{S}_z - \frac{\gamma B_1}{\hbar}\hat{S}_x\right]}|jm\rangle\right|^2$$

$$= \delta_{jj'}\left|\langle jm'|\, e^{-it\left[(\omega-\Omega)\hat{S}_z - \frac{\gamma B_1}{\hbar}\hat{S}_x\right]}|jm\rangle\right|^2$$

since $\hat{S}^2$ commutes with $e^{i(\alpha\hat{S}_z + \beta\hat{S}_x)}$ and $|e^{i\omega t m'}|^2 = 1$

calculating this result is complicated.

example: spin $\frac{1}{2}$

$$\hat{S}_z = \tfrac{1}{2}\sigma_z = \tfrac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \hat{S}_x = \tfrac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \tfrac{1}{2}\sigma_x$$

$$e^{i\vec{v}\cdot\vec{\sigma}} = \cos|v| + i\sin|v|\,\frac{\vec{v}\cdot\vec{\sigma}}{|v|}$$ derived earlier in class

so $e^{-it[(\omega-\Omega)\hat{S}_z - \frac{\gamma B_1}{\hbar}\hat{S}_x]} = e^{i\vec{v}\cdot\vec{\sigma}}$

$|v| = \frac{t}{2}\sqrt{(\omega-\Omega)^2 + (\frac{\gamma B_1}{\hbar})^2} = \Delta \frac{t}{2}$ $\quad \Delta = \sqrt{(\omega-\Omega)^2 + (\frac{\gamma B_1}{\hbar})^2}$

$$e^{-it[(\omega-\Omega)\hat{S}_z - \frac{\gamma B_1}{\hbar}\hat{S}_x]}$$

$$= \cos\frac{\Delta t}{2}\,\mathbb{1} - i\sin\frac{\Delta t}{2}\left(\frac{(\omega-\Omega)}{\Delta}\sigma_z - \frac{\gamma B_1}{\hbar\Delta}\sigma_x\right)$$

and

$$\left|\langle \tfrac{1}{2}, -\tfrac{1}{2}|\cos\frac{\Delta t}{2}\,\mathbb{1} - i\sin\frac{\Delta t}{2}\left(\frac{\omega-\Omega}{\Delta}\sigma_z - \frac{\gamma B_1}{\hbar\Delta}\sigma_x\right)|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right|^2$$

$$= \sin^2\frac{\Delta t}{2}\,\frac{\gamma^2 B_1^2}{\hbar^2\Delta^2} = P_{-\frac{1}{2}\leftarrow\frac{1}{2}}(t)$$

$$P_{\frac{1}{2}\leftarrow\frac{1}{2}}(t) = \left|\cos\frac{\Delta t}{2} - i\sin\frac{\Delta t}{2}\,\frac{\omega-\Omega}{\Delta}\right|^2$$

$$= \cos^2\frac{\Delta t}{2} + \sin^2\frac{\Delta t}{2}\left(\frac{\omega-\Omega}{\Delta}\right)^2$$

$$= 1 - \left(\frac{\gamma B_1}{\Delta\hbar}\right)^2\sin^2\frac{\Delta t}{2}$$

Note $P_{\frac{1}{2}\leftarrow\frac{1}{2}}(t) + P_{-\frac{1}{2}\leftarrow\frac{1}{2}}(t) = 1$ as it must.