Driven Simple Harmonic Oscillator

an exact time-ordered product.

$$\hat{H} = \hat{H}_0 + \hat{V}(t) \qquad \hat{H}_0 = \hbar\omega(a^\dagger a + \tfrac{1}{2}) \qquad \hat{V}(t) = f(t)\hat{a}^\dagger + f^*(t)\hat{a}$$

$$(a, a^\dagger)_- = 1$$

since $a + a^\dagger \sim x$ this acts like a driving force moving x as a function of time.

Define $\hat{A} = \hat{a} + \dfrac{f(t)}{\hbar\omega}$

$$(\hat{A}, \hat{A}^\dagger)_- = (\hat{a}, \hat{a}^\dagger)_- = 1$$

$$\hbar\omega(\hat{A}^\dagger\hat{A} + \tfrac{1}{2}) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\frac{f(t)}{\hbar\omega} + \hat{a}\frac{f^*(t)}{\hbar\omega} + \frac{|f(t)|^2}{(\hbar\omega)^2} + \tfrac{1}{2}\right)$$

$$= \hat{H}(t) + \frac{|f(t)|^2}{\hbar\omega}$$

so $\hat{H}(t) = \hbar\omega(\hat{A}^\dagger\hat{A} + \tfrac{1}{2}) - \dfrac{|f(t)|^2}{\hbar\omega}$

define $\hat{A}|0\rangle = 0$

$$|n\rangle = \frac{(\hat{A}^\dagger)^n}{\sqrt{n!}}|0\rangle$$

$$\hat{H}|n\rangle = E_n(t)|n\rangle = \left[\hbar\omega(n + \tfrac{1}{2}) - \frac{|f(t)|^2}{\hbar\omega}\right]|n\rangle$$

expand $|\psi(t)\rangle = \sum_n c_n(t)|n\rangle$

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \sum_n i\hbar\left[\frac{\partial c_n(t)}{\partial t}|n\rangle + c_n(t)\frac{\partial}{\partial t}|n\rangle\right] = \hat{H}|\psi(t)\rangle$$

$$= \sum_n c_n(t)E_n(t)|n\rangle$$

multiply by $\langle m|$ to get

$$i\hbar \frac{\partial C_m(t)}{\partial t} + i\hbar \sum_n C_n(t) \langle m| \frac{\partial}{\partial t} |n\rangle = E_m(t)\, C_m(t)$$

$$\frac{\partial}{\partial t}|n\rangle = \frac{\partial}{\partial t} \frac{(\hat{A}^+)^n}{\sqrt{n!}} |0\rangle = \frac{\partial}{\partial t} \frac{\left(\hat{a}^+ + \frac{f^*(t)}{\hbar\omega}\right)^n}{\sqrt{n!}} |0\rangle$$

$$= n \frac{\left(\hat{a}^+ + \frac{f^*(t)}{\hbar\omega}\right)^{n-1}}{\sqrt{n!}} \frac{df^*}{dt} \frac{1}{\hbar\omega} |0\rangle + \frac{(\hat{A}^+)^n}{\sqrt{n!}} \frac{\partial}{\partial t} |0\rangle$$

$$= \frac{\sqrt{n}}{\hbar\omega} \frac{df^*}{dt} |n-1\rangle + \frac{(\hat{A}^+)^n}{\sqrt{n!}} \frac{\partial}{\partial t} |0\rangle$$

but $\hat{A}|0\rangle = 0 \Rightarrow \frac{\partial}{\partial t} \left(\hat{a} + \frac{f(t)}{\hbar\omega}\right) |0\rangle = 0$

$$= \frac{df}{dt} \frac{1}{\hbar\omega} |0\rangle + \left(\hat{a} + \frac{f(t)}{\hbar\omega}\right) \frac{\partial}{\partial t} |0\rangle = 0$$

multiply by $\langle m|$ to get $\frac{df(t)}{dt} \frac{1}{\hbar\omega} \delta_{m0} + \langle m| \hat{A} \frac{\partial}{\partial t} |0\rangle = 0$

$$\sqrt{m+1} \langle m+1| \frac{\partial}{\partial t} |0\rangle = - \frac{df(t)}{dt} \frac{1}{\hbar\omega} \delta_{m0}$$

so $\langle m| \frac{\partial}{\partial t} |0\rangle = - \frac{df(t)}{dt} \frac{1}{\hbar\omega} \delta_{m1}$

$$\Rightarrow \frac{\partial}{\partial t} |0\rangle = - \frac{df(t)}{dt} \frac{1}{\hbar\omega} |1\rangle$$

so $\frac{\partial}{\partial t} |n\rangle = \frac{\sqrt{n}}{\hbar\omega} \frac{df^*(t)}{dt} |n-1\rangle - \frac{\sqrt{n+1}}{\hbar\omega} \frac{df(t)}{dt} |n+1\rangle$

substitute into equation above

$$i\hbar \frac{\partial C_m(t)}{\partial t} + i\hbar \frac{\sqrt{m+1}}{\hbar\omega} \frac{df^*(t)}{dt} C_{m+1}(t) - i\hbar \frac{\sqrt{m}}{\hbar\omega} \frac{df(t)}{dt} C_{m-1}(t) = E_m(t)\, C_m(t)$$

So we have

$$\boxed{\frac{dC_m(t)}{dt} = -\frac{i}{\hbar} E_m(t)\, C_m(t) - \frac{\sqrt{m+1}}{\hbar\omega}\frac{df^*(t)}{dt} C_{m+1}(t) + \frac{\sqrt{m}}{\hbar\omega}\frac{df(t)}{dt} C_{m-1}(t)}$$

This equation is a complicate coupled linear differential equation with no obvious solution when $f \neq 0$

So, let's examine instead with the interaction representation picture.

$$|\psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_0 t}\, T \exp\left[-\frac{i}{\hbar}\int_0^t dt'\, \hat{V}_I(t')\right]|\psi_S(0)\rangle$$

$$\hat{V}_I(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}\,\hat{V}(t)\, e^{-\frac{i}{\hbar}\hat{H}_0 t}$$

In our case, we have

$$\hat{V}_I(t) = e^{\frac{i}{\hbar}\hbar\omega \hat{a}^+\hat{a} t}\left[f\hat{a}^+ + f^*\hat{a}\right] e^{-i\omega a^+ a t}$$

But $$e^{i\lambda \hat{a}^+\hat{a}}\,\hat{a}^+\, e^{-i\lambda\hat{a}^+\hat{a}} = g(\lambda)$$

$$i\, e^{i\lambda\hat{a}^+\hat{a}}\left[\hat{a}^+\hat{a}, \hat{a}^+\right] e^{-i\lambda\hat{a}^+\hat{a}} = \frac{d\,g(\lambda)}{d\lambda}$$

$$i\, g(\lambda) = \frac{dg(\lambda)}{d\lambda} \Rightarrow g(\lambda) = g(0)\, e^{i\lambda}$$

so $$\boxed{\hat{V}_I(t) = \left[f(t)\, e^{i\omega t}\,\hat{a}^+ + f^*(t)\, e^{-i\omega t}\,\hat{a}\right]}$$

Note that

$$[\hat{V}_I(t), \hat{V}_I(t')] = f(t) f^*(t') e^{i\omega t} e^{-i\omega t'} [\hat{a}^+, \hat{a}]$$
$$+ f^*(t) f(t') e^{-i\omega t} e^{i\omega t'} [\hat{a}, \hat{a}^+]$$
$$= -2i \, \mathrm{Im}\left[f(t) f^*(t') e^{i\omega(t-t')}\right]$$
$$= \text{number, not an operator.}$$

so $[\hat{V}_I(t), \hat{V}_I(t')]$ commutes with all operators.

To be concrete, choose $f(t) = c e^{i\Omega t}$ $\quad c = \text{real}$

$$\hat{V}_I(t) = c\left(e^{i(\omega+\Omega)t} \hat{a}^+ + e^{-i(\omega+\Omega)t} \hat{a}\right)$$
$$[\hat{V}_I(t), \hat{V}_I(t')] = -2ic^2 \, \mathrm{Im}\left[e^{i\Omega(t-t')} e^{i\omega(t-t')}\right]$$
$$= -2ic^2 \sin\left[(\omega+\Omega)(t-t')\right]$$

Gottfried says to consider

$$\hat{w}(t) = \int_0^t \hat{V}_I(t')\,dt' = \frac{c}{i(\omega+\Omega)}\left(e^{i(\omega+\Omega)t} - 1\right)\hat{a}^+$$
$$- \frac{c}{i(\omega+\Omega)}\left(e^{-i(\omega+\Omega)t} - 1\right)\hat{a}$$

Then

$$[\hat{w}(t), \hat{V}_I(t)] = \frac{c^2}{i(\omega+\Omega)}\left(1 - e^{-i(\omega+\Omega)t}\right)(-1) - \frac{c^2}{i(\omega+\Omega)}\left(1 - e^{i(\omega+\Omega)t}\right)$$
$$= \frac{-2c^2}{i(\omega+\Omega)}\left(1 - \cos(\omega+\Omega)t\right) = \text{number again}$$

Now make the unitary transformation $|\psi_I(t)\rangle = e^{-\frac{i}{\hbar}\hat{w}(t)} |\psi_{II}(t)\rangle$

but $\left[i\hbar \frac{\partial}{\partial t} - \hat{V}_I(t)\right] |\psi_I(t)\rangle = 0$

so we have

$$e^{\frac{i}{\hbar}\hat{W}(t)}\left[i\hbar\frac{\partial}{\partial t}-\hat{V}_I(t)\right]\overbrace{e^{-\frac{i}{\hbar}\hat{W}(t)}\underbrace{e^{\frac{i}{\hbar}\hat{W}(t)}|\psi_I(t)\rangle}_{|\psi_{II}(t)\rangle}}^{\mathbb{1}}=0$$

so $e^{\frac{i}{\hbar}\hat{W}(t)}\left[i\hbar\frac{\partial}{\partial t}-\hat{V}_I(t)\right]e^{-\frac{i}{\hbar}\hat{W}(t)}|\psi_{II}(t)\rangle=0$

Since $\hat{W}(t)$ does not commute with $\hat{V}_I(t)$, we cannot easily evaluate the derivative term. So let us expand the exponentials in a power series and then differentiate.

acts on everything to the right

$$\left(1+\frac{i}{\hbar}\hat{W}+\frac{1}{2!}\left(\frac{i}{\hbar}\right)^2\hat{W}^2+\frac{1}{3!}\left(\frac{i}{\hbar}\right)^3\hat{W}^3+\dots\right)\left[i\hbar\frac{\partial}{\partial t}-\hat{V}_I\right]\left(1-\frac{i}{\hbar}\hat{W}+\frac{1}{2!}\left(\frac{-i}{\hbar}\right)^2\hat{W}^2+\frac{1}{3!}\left(\frac{-i}{\hbar}\right)^3\hat{W}^3+\dots\right)$$

$$\left(1+\frac{i}{\hbar}\hat{W}+\frac{1}{2!}\left(\frac{i}{\hbar}\right)^2\hat{W}^2+\frac{1}{3!}\left(\frac{i}{\hbar}\right)^3\hat{W}^3+\dots\right)\Big\{i\hbar\frac{\partial}{\partial t}-\hat{V}_I-\frac{i}{\hbar}\left(i\hbar\dot{\hat{W}}+\hat{W}i\hbar\frac{\partial}{\partial t}-\hat{V}_I\hat{W}\right)$$
$$+\frac{1}{2!}\left(\frac{-i}{\hbar}\right)^2\left[i\hbar\dot{\hat{W}}\hat{W}+i\hbar\hat{W}\dot{\hat{W}}+\hat{W}^2i\hbar\frac{\partial}{\partial t}-\hat{V}_I\hat{W}^2\right]$$
$$+\frac{1}{3!}\left(\frac{-i}{\hbar}\right)^3\left[i\hbar\dot{\hat{W}}\hat{W}^2+i\hbar\hat{W}\dot{\hat{W}}\hat{W}+i\hbar\hat{W}^2\dot{\hat{W}}+\hat{W}^3i\hbar\frac{\partial}{\partial t}-\hat{V}_I\hat{W}^3\right]+\dots\Big\}$$

$$=i\hbar\frac{\partial}{\partial t}-\hat{V}_I+\dot{\hat{W}}+\hat{W}\frac{\partial}{\partial t}+\frac{i}{\hbar}\hat{V}_I\hat{W}-\hat{W}\frac{\partial}{\partial t}-\frac{i}{\hbar}\hat{W}\hat{V}_I$$
$$-\frac{1}{2}\frac{i}{\hbar}\left(\dot{\hat{W}}\hat{W}+\hat{W}\dot{\hat{W}}+\hat{W}^2\frac{\partial}{\partial t}-\frac{i}{\hbar}\hat{V}_I\hat{W}^2\right)+\frac{i}{\hbar}\left(\hat{W}\dot{\hat{W}}+\hat{W}^2\frac{\partial}{\partial t}+\frac{i}{\hbar}\hat{W}\hat{V}_I\hat{W}\right)$$
$$-\frac{1}{2}\frac{i}{\hbar}\hat{W}^2\frac{\partial}{\partial t}+\frac{1}{2}\frac{1}{\hbar^2}\hat{W}^2\hat{V}_I+\frac{1}{6}\left(\frac{i}{\hbar}\right)^2\left(\dot{\hat{W}}\hat{W}^2+\hat{W}\dot{\hat{W}}\hat{W}+\hat{W}^2\dot{\hat{W}}\right)-\frac{1}{2}\left(\frac{i}{\hbar}\right)^2$$
$$*\left(\hat{W}\dot{\hat{W}}\hat{W}+\hat{W}^2\dot{\hat{W}}\right)+\frac{1}{2}\left(\frac{i}{\hbar}\right)^2*\left(\hat{W}^2\dot{\hat{W}}\right)$$
$$+\text{higher order}$$

But $\dot{\hat{W}}=\frac{d}{dt}\hat{W}(t)=\hat{V}_I(t)$ so we get

$$= i\hbar\frac{\partial}{\partial t} - \hat{V}_I + \hat{V}_I - \frac{i}{\hbar}[\hat{W},\hat{V}_I] - \frac{i}{2\hbar}[\hat{V}_I\hat{W} + \hat{W}\hat{V}_I - 2\hat{W}\hat{V}_I]$$

$$+ \frac{1}{2\hbar^2}\hat{V}_I\hat{W}^2 - \frac{1}{\hbar^2}\hat{W}\hat{V}_I\hat{W} + \frac{1}{2\hbar^2}\hat{W}^2\hat{V}_I - \frac{1}{6\hbar^2}(\hat{V}_I\hat{W}^2 + \hat{W}\hat{V}_I\hat{W} + \hat{W}^2\hat{V}_I)$$

$$+ \frac{1}{2\hbar^2}(\hat{W}\hat{V}_I\hat{W} + \hat{W}^2\hat{V}_I) - \frac{1}{2\hbar^2}\hat{W}^2\hat{V}_I + \text{higher order}$$

$$= i\hbar\frac{\partial}{\partial t} - \frac{i}{2\hbar}[\hat{W},\hat{V}_I] + \frac{1}{\hbar^2}\left[+\frac{1}{3}\hat{V}_I\hat{W}^2 - \frac{2}{3}\hat{W}\hat{V}_I\hat{W} + \frac{1}{3}\hat{W}^2\hat{V}_I\right] + \cdots$$

$$= i\hbar\frac{\partial}{\partial t} - \frac{i}{2\hbar}[\hat{W},\hat{V}_I] + \frac{1}{3\hbar^2}\left[-[\hat{W},\hat{V}_I]\hat{W} + \hat{W}[\hat{W},\hat{V}_I]\right] + \cdots$$

$$= i\hbar\frac{\partial}{\partial t} - \frac{i}{2\hbar}[\hat{W},\hat{V}_I] + \frac{1}{3\hbar^2}[\hat{W},[\hat{W},\hat{V}_I]] + \cdots$$

all higher order terms are multiple commutators

But $[\hat{W},\hat{V}_I]$ commutes with everything, so

$$= i\hbar\frac{\partial}{\partial t} - \frac{i}{2\hbar}[\hat{W},\hat{V}_I] = i\hbar\frac{\partial}{\partial t} + \frac{c^2}{\hbar(\omega+\Omega)}(1-\cos(\omega+\Omega)t)$$

so $$\left[i\hbar\frac{\partial}{\partial t} + \frac{c^2}{\hbar(\omega+\Omega)}[1-\cos(\omega+\Omega)t]\right]|\psi_{II}(t)\rangle = 0$$

$$\Rightarrow |\psi_{II}(t)\rangle = e^{\frac{i}{\hbar}\int_0^t \frac{c^2}{\hbar(\omega+\Omega)}[1-\cos(\omega+\Omega)t']dt'}|\psi_{II}(0)\rangle$$

$$= e^{\frac{ic^2}{\hbar^2(\omega+\Omega)}\left[t - \frac{\sin(\omega+\Omega)t}{\omega+\Omega}\right]}|\psi_{II}(0)\rangle$$

Hence

$$|\psi_I(t)\rangle = e^{-\frac{i}{\hbar}\hat{W}(t)} |\psi_{II}(t)\rangle$$

$$= e^{-\frac{c}{\hbar(\omega+\Omega)}(e^{i(\omega+\Omega)t}-1)\hat{a}^+ + \frac{c}{\hbar(\omega+\Omega)}(e^{-i(\omega+\Omega)t}-1)\hat{a}}$$

$$* \; e^{\frac{ic^2}{\hbar^2(\omega+\Omega)}\left[t - \frac{\sin(\omega+\Omega)t}{(\omega+\Omega)}\right]} |\psi_{II}(0)\rangle$$

and $|\psi_S(t)\rangle = e^{-i\omega t(\hat{a}^+\hat{a}+\frac{1}{2})} |\psi_I(t)\rangle$

To check, we could compute overlaps with states $\langle m|$ and verify the differential equation for $C_m(t)$ holds, but I won't do that.

In summary, when $[\hat{V}_I(t), \hat{V}_I(t')]$ = number not an operator the time ordered product simplifies.
In particular, we have

$$\boxed{\begin{aligned} T\exp\left[-\frac{i}{\hbar}\int_0^t \hat{V}_I(t')dt'\right] &= T\exp\left\{-\frac{i}{\hbar}c\int_0^t \left[e^{i(\omega+\Omega)t'}\hat{a}^+ + e^{-i(\omega+\Omega)t'}\hat{a}\right]dt'\right. \\ &= \exp\left[-\frac{c}{\hbar(\omega+\Omega)}(e^{i(\omega+\Omega)t}-1)\hat{a}^+ + \frac{c}{\hbar(\omega+\Omega)}(e^{-i(\omega+\Omega)t}-1)\hat{a}\right] \\ &\quad * \exp\left[\frac{ic^2}{\hbar^2(\omega+\Omega)^2}[(\omega+\Omega)t - \sin(\omega+\Omega)t]\right] \end{aligned}}$$

for this case.