Time dependent perturbation theory.

Our general interaction picture formalism showed

$$|\psi_s(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)} \hat{U}_I(t,t_0)|\psi_s(t_0)\rangle$$

$$\hat{U}_I(t,t_0) = T e^{-\frac{i}{\hbar}\int_{t_0}^{t}\hat{V}_I(t')dt'} = 1 - \frac{i}{\hbar}\int_{t_0}^{t}\hat{V}_I(t')dt' + \left(\frac{-i}{\hbar}\right)^2\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2 \, \hat{V}_I(t_1)\hat{V}_I(t_2) + \dots$$

Then the probability to have a transition from state $|i\rangle$ at t_0 to state $\langle f|$ at time t is

$$P_{f\leftarrow i}(t) = \left|\langle f| e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\hat{U}_I(t,t_0)|i\rangle\right|^2$$

If ~~[illegible]~~ $\langle f|$ and $|i\rangle$ are eigenstates of $\hat{H}_0$ then

$\langle f| e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)} = e^{-\frac{i}{\hbar}E_f(t-t_0)}\langle f|$ is a phase whose mod $=1$

so

$$P_{n\leftarrow m}(t) = \left|{}_0\langle n|\hat{U}_I(t,t_0)|m\rangle_0\right|^2$$

$$= \left|{}_0\langle n| 1 - \frac{i}{\hbar}\int_0^t dt'\,\hat{V}_I(t') + \dots |m\rangle_0\right|^2$$

$$= \left|\delta_{nm} - \frac{i}{\hbar}\,{}_0\langle n|\int_{t_0}^{t} e^{\frac{i}{\hbar}\hat{H}_0(t_1-t_0)}\hat{V}(t_1)e^{-\frac{i}{\hbar}\hat{H}_0(t_1-t_0)}|m\rangle_0 + \dots\right|^2$$

define $\omega_{nm} = (E_n^0 - E_m^0)/\hbar$ then the lowest order approximation, called the first Born approximation is

$$P_{n\leftarrow m}(t) \sim \left|\delta_{nm} - \frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{nm}(t_1-t_0)}\,{}_0\langle n|\hat{V}(t_1)|m\rangle_0\, dt_1\right|^2$$

$$= \left|\delta_{nm} - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, e^{i\omega_{nm}(t_1-t_0)}V_{nm}(t_1)\right|^2$$

so if $n \neq m$

$$\boxed{P_{\substack{n \leftarrow m \\ n\neq m}}(t) = \frac{1}{\hbar^2} \left| \int_{t_0}^{t} dt_1 \, e^{i\omega_{nm}(t_1 - t_0)} V_{nm}(t_1) \right|^2}$$

Harmonic perturbation

$$\hat{V}(t) = e^{i\Omega t} \hat{a}^{+} + e^{-i\Omega t} \hat{a} \qquad \Omega > 0 = \text{driving frequency}$$

$\hat{a}$ is any operator

Assume $a_{nm} = {}_0\langle n | \hat{a} | m \rangle_0 \neq 0$ and $t_0 = 0$, then

$$P_{\substack{n \leftarrow m \\ n\neq m}}(t) = \frac{1}{\hbar^2} \left| \int_0^t dt_1 \, e^{i\omega_{nm} t_1} \left(a_{nm} e^{-i\Omega t_1} + a_{nm}^{*} e^{i\Omega t_1} \right) \right|^2$$

$$= \frac{1}{\hbar^2} \left| \frac{a_{nm}}{i(\omega_{nm} - \Omega)} \left(e^{i(\omega_{nm} - \Omega)t} - 1 \right) + \frac{a_{nm}^{*}}{i(\omega_{nm} + \Omega)} \left(e^{i(\omega_{nm} + \Omega)t} - 1 \right) \right|^2$$

$$= \frac{1}{\hbar^2} \left\{ \frac{|a_{nm}|^2}{(\omega_{nm} - \Omega)^2} 2\left(1 - \cos(\omega_{nm} - \Omega)t\right) \right.$$

$$\left. + \frac{|a_{nm}|^2}{(\omega_{nm} + \Omega)^2} 2\left(1 - \cos(\omega_{nm} + \Omega)t\right) + \text{cross terms} \right\}$$

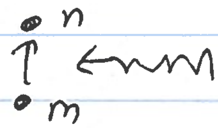
$$= \frac{4}{\hbar^2} |a_{nm}|^2 \left[\frac{\sin^2 (\omega_{nm} - \Omega)\frac{t}{2}}{(\omega_{nm} - \Omega)^2} + \frac{\sin^2 (\omega_{nm} + \Omega)t/2}{(\omega_{nm} + \Omega)^2} + \text{cross} \right]$$

large if $\omega_{nm} > 0$ and $\omega_{nm} \sim \Omega$

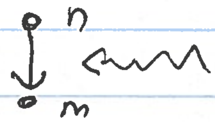
large if $\omega_{nm} < 0$ and $\omega_{nm} \sim -\Omega$

Both conditions are called resonance.

$\omega_{nm} \to \Omega$ $\quad E_n = E_m + \hbar\Omega$ $\quad$ stimulated absorption



$-\omega_{nm} \to \Omega \;\to\; E_n = E_m - \hbar\Omega$ $\quad$ stimulated emission



Accuracy expect first Born to be accurate

for $P_{n\leftarrow m} \ll 1$, Worst case is on resonance
where $P_{n\leftarrow m} \sim ct^2$ which is larger than 1
for some time
In general probabilities oscillate with time
(recall cyclotron resonance problem)

The problem with first order perturbation theory is it
neglects depletion and return

depletion: expect probability of $P_{n\leftarrow m}$ to decrease
when most m's are gone

return: after n's populated, they re-emit back to m.

Both processes are higher order effects.

example photo ionization of Hydrogen – when a photon knocks
an electron out of H, little chance it will return back. In this
case, neglecting return is OK.

Compare to the solvable example of last lecture.

recall we showed

$$|\psi_s(t)\rangle = e^{i\omega t(\hat a^\dagger \hat a + \frac{1}{2})} \exp\left[\frac{c}{\hbar(\Omega+\omega)}\left(e^{i(\omega+\Omega)t}-1\right)\hat a^\dagger + \frac{c}{\hbar(\omega+\Omega)}\left(e^{-i(\omega+\Omega)t}-1\right)\hat a \right]$$

$$* \quad e^{-i\frac{c^2}{\hbar^2(\Omega+\omega)^2}\left[(\omega+\Omega)t - \sin(\omega+\Omega)t\right]} |\psi_s(0)\rangle$$

for $\hat H = \underbrace{\hbar\omega(\hat a^\dagger \hat a + \frac{1}{2})}_{\hat H_0} + \underbrace{c e^{i\Omega t}\hat a^\dagger + c e^{-i\Omega t}\hat a}_{\hat V}$ $\qquad c = \text{real}$

consider the following operator identity.

$$e^{\tau(\hat A+\hat B)} e^{-\tau\hat B} e^{-\tau\hat A} = f(\tau) \qquad \text{with } [\hat A,\hat B] = \text{number}$$

$$e^{\tau(\hat A+\hat B)}(\hat A+\hat B)e^{-\tau\hat B}e^{-\tau\hat A} - e^{\tau(\hat A+\hat B)}\hat B e^{-\tau\hat B}e^{-\tau\hat A} - e^{\tau(\hat A+\hat B)}e^{-\tau\hat B}\hat A e^{-\tau\hat A} = \frac{df(\tau)}{d\tau}$$

so $\quad e^{\tau(\hat A+\hat B)}\left[\hat A, e^{-\tau\hat B}\right]e^{-\tau\hat A} = \frac{df(\tau)}{d\tau}$

But $\left[\hat A, e^{-\tau\hat B}\right] = \sum_{n=0}^{\infty}\frac{(-\tau)^n}{n!}\left[\hat A,\hat B^n\right] = \sum_{n=0}^{\infty}\frac{(-\tau)^n}{n!}[\hat A,\hat B]\hat B^{n-1}\cdot n$

when $[\hat A,\hat B] = \text{number}$

$$= -\tau[\hat A,\hat B]e^{-\tau\hat B}$$

so $\frac{df(\tau)}{d\tau} = \tau[\hat B,\hat A]f(\tau) \Rightarrow f(\tau) = e^{\frac{\tau^2}{2}[\hat B,\hat A]}$ $\qquad$ let $\tau=1$

or $e^{\hat A+\hat B} = e^{\frac{1}{2}[\hat B,\hat A]}e^{A}e^{B}$

apply to $\hat{A} = -\frac{c}{\hbar(\omega+\Omega)}\left(e^{i(\omega+\Omega)t}-1\right)\hat{a}^{+}$

$$\hat{B} = \frac{c}{\hbar(\omega+\Omega)}\left(e^{-i(\omega+\Omega)t}-1\right)\hat{a}$$

$$[\hat{B},\hat{A}] = \frac{-c^2}{\hbar^2(\omega+\Omega)^2}\,2\left(1-\cos(\omega+\Omega)t\right)$$

Thus, if we start in the ground state, we find

$$P_{m\leftarrow 0}(t) = \Big|\, {}_0\langle m|\, \overset{\swarrow \text{phase only}}{e^{i\omega t(\hat{a}^{+}\hat{a}+\frac{1}{2})}}\, e^{-\frac{c}{\hbar(\omega+\Omega)}\left(e^{i(\omega+\Omega)t}-1\right)\hat{a}^{+}}$$

$$e^{\frac{c}{\hbar(\omega+\Omega)}\left(e^{-i(\omega+\Omega)t}-1\right)\hat{a}}\,|0\rangle_0 \quad \leftarrow \hat{a}|0\rangle = 0$$

$$* \; e^{-\frac{c^2}{\hbar^2(\omega+\Omega)^2}\left(1-\cos(\omega+\Omega)t\right)}\, \underbrace{e^{i\frac{c^2}{\hbar^2(\omega+\Omega)^2}\left((\omega+\Omega)t-\sin(\omega+\Omega)t\right)}}_{\text{phase only}}\Big|^2$$

$$= e^{-\frac{2c^2}{\hbar^2(\omega+\Omega)^2}\left(1-\cos(\omega+\Omega)t\right)}\,\Big|\,{}_0\langle m|\, e^{-\frac{c}{\hbar(\omega+\Omega)}\left(e^{i(\omega+\Omega)t}-1\right)\hat{a}^{+}}|0\rangle_0\Big|^2$$

but ${}_0\langle m| = {}_0\langle 0|\frac{(\hat{a})^m}{\sqrt{m!}}$ and $e^{-\frac{c}{\hbar(\omega+\Omega)}\left(e^{i(\omega+\Omega)t}-1\right)\hat{a}^{+}}$

$$= \sum_{n=0}^{\infty}\left(\frac{-c}{\hbar(\omega+\Omega)}\left(e^{i(\omega+\Omega)t}-1\right)\right)^n \frac{1}{n!}(\hat{a}^{+})^n$$

need $n=m$ and $\langle 0|(\hat{a})^m(\hat{a}^{+})^m|0\rangle = m!$ so

$$P_{m\leftarrow 0}(t) = e^{-\frac{2c^2}{\hbar^2(\omega+\Omega)^2}\left(1-\cos(\omega+\Omega)t\right)}\left[\frac{c^2}{\hbar^2(\omega+\Omega)^2}2\left(1-\cos(\omega+\Omega)t\right)\right]^m\frac{1}{m!}$$

$$\boxed{P_{m\leftarrow 0}(t) = \frac{1}{m!}\left(\frac{2c^2}{\hbar^2(\omega+\Omega)^2}\right)^m\left(1-\cos(\omega+\Omega)t\right)^m e^{-\frac{2c^2}{\hbar^2(\omega+\Omega)^2}\left(1-\cos(\omega+\Omega)t\right)}}$$

This is the exact solution

One can directly check that

$$\sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{2c^2}{\hbar^2(\omega+\Omega)^2} \right)^m (1-\cos(\omega+\Omega)t)^m \, e^{-\frac{2c^2}{\hbar^2(\omega+\Omega)^2}(1-\cos(\omega+\Omega)t)}$$

$$= \exp\left[\frac{2c^2}{\hbar^2(\omega+\Omega)^2}(1-\cos(\omega+\Omega)t) - \frac{2c^2}{\hbar^2(\omega+\Omega)^2}(1-\cos(\omega+\Omega)t) \right]$$

$= 1$ as it must.

Compare to the harmonic calculation

$\langle m | c\hat{a}^{+} | 0 \rangle = 0$ unless $m = 1$

$\langle m | c\hat{a}^{+} | 0 \rangle = c\,\delta_{m1}$ $\qquad \omega_{10} = \omega$

$$\boxed{P_{1\leftarrow 0}(t) \cong \frac{4c^2}{\hbar^2} \frac{\sin^2 (\omega+\Omega)t\frac{1}{2}}{(\omega+\Omega)^2}}$$

which agrees with the above form for $m=1$ to lowest order in c^2 when we note that

$1-\cos(\omega+\Omega)t = 2\sin^2[(\omega+\Omega)t/2]$