Ion traps and quantum simulation

Ions are trapped in a Linear Paul trap via harmonic confinement plus Coulomb repulsion

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i>j}^{N} \frac{e^2}{|r_i - r_j|} + \sum_{i=1}^{N}\left(\frac{1}{2} m \omega_x^2 r_{ix}^2 + \frac{1}{2} m\omega_y^2 r_{iy}^2 + \frac{1}{2} m\omega_z^2 r_{iz}^2\right)$$

KE Coulomb trap

Use classical mechanics to solve for equilibrium positions and for harmonic deviations about equilibrium to find

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{\substack{i=1\\j=1}}^{N}\left(\frac{1}{2} r_{ix} K_{ij}^{x} r_{jx} + \frac{1}{2} r_{iy} K_{ij}^{y} r_{jy} + \frac{1}{2} r_{iz} K_{ij}^{z} r_{iz}\right)$$

where K_{ij} is the spring constant matrix.

Introduce normal modes b_i^{α} which are eigenvectors of the K matrices

$$\sum_j K_{ij}^{x} b_j = k_\alpha^x b_i \quad \text{and similar for } y \text{ and } z$$

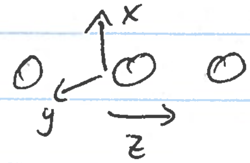
and define frequencies via $\omega_\alpha^x = \sqrt{\frac{k_\alpha^x}{m}}$ and similar for y & z

Then the quantized phonon Hamiltonian is

$$H_{phonon} = \sum_{\substack{j=x,\\y,z}} \sum_{\alpha=1}^{N} \hbar \omega_\alpha^j \left(a_{\alpha j}^{\dagger} a_{\alpha j} + \frac{1}{2}\right)$$

where $a_{\alpha j}$ is the phonon destruction operator in the αth normal mode for direction j.

Example: 3 site chain.

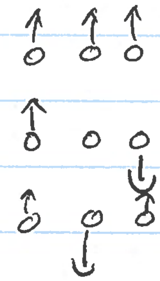


the transverse normal modes are

center of mass $\vec{b}^{com} = \frac{1}{\sqrt{3}}(1,1,1)$

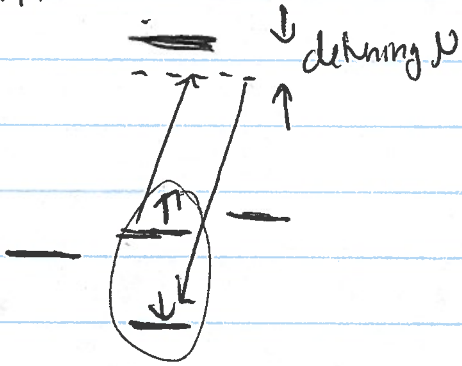
tilt $\vec{b}^{tilt} = \frac{1}{\sqrt{2}}(1,0,-1)$

zig zag $\vec{b}^{zigzag} = \frac{1}{\sqrt{6}}(1,-2,1)$



In general we have ~~three~~ N phonon modes for each direction.

Each ion has a complicated level structure. The Yb^+_{171} ion looks like



shine two lasers on detuned from the excited state but high power to drive the transition from the state we call up to the one we call down.

Clock states have no z component of total angular momentum so no linear Zeeman shift.

This looks like a $(\sigma_+ + \sigma_-)$ interaction since $\uparrow \to \downarrow \to \uparrow$ periodically. (Ω_j = Rabi frequency = rate of flips)

But $\sigma_+ + \sigma_- \sim \sigma^x_j$ so we have found a σ^x_j operation on our ions.

The ~~tacee~~ ions also couple to the spatial profile of the laser beam. ~~By~~ Because we have two beams at different frequencies and different k vectors, we find the laser ion interaction is

$$H_{laser\text{-}ion} = -\sum_{j=1}^{N} \underbrace{\hbar\Omega_j}_{\text{Rabi freq.}} \left(\underbrace{\delta\vec{k}}_{k_1 - k_2 \text{ of laser beams}} \cdot \underbrace{\delta\vec{R}_j}_{\text{position of jth ion relative to equilibrium}}\right) \underbrace{\sigma_j^x}_{\text{pauli spin matrix at jth location}} \underbrace{\sin\mu t}_{\text{oscillation from detuning}}$$

We write $\delta\vec{R}_j = \sum_{\alpha=1}^{N} b_j^{\alpha x} \sqrt{\frac{\hbar}{2m\omega_\alpha^x}} \left(a_\alpha^{+x} + a_\alpha^x\right) + y \;\&\; z \text{ terms}$

For experiments $\delta\vec{k}$ is in the x direction only.

So we set (focus only on x states)

$$H = \underbrace{\sum_{\alpha=1}^{N} \hbar\omega_\alpha\left(a_\alpha^+ a_\alpha + \tfrac{1}{2}\right)}_{H_0} - \underbrace{\sum_{j=1}^{N} \hbar\Omega_j\, \delta k \sum_{\alpha=1}^{N} \sqrt{\frac{\hbar}{2m\omega_\alpha}} \left(a_\alpha^+ + a_\alpha\right) b_j^\alpha \sigma_j^x \sin\mu t}_{V(t)}$$

Go to the interaction representation and note

$$e^{\frac{iH_0 t}{\hbar}} a_\alpha^+ e^{-\frac{iH_0 t}{\hbar}} = e^{i\omega_\alpha t} a_\alpha^+$$

$$e^{\frac{iH_0 t}{\hbar}} a_\alpha e^{-\frac{iH_0 t}{\hbar}} = e^{-i\omega_\alpha t} a_\alpha$$

We need to add a transverse magnetic field as well

$$H_B(t) = +\sum_{j=1}^{N} B(t)\, \sigma_j^y$$

which will be our control parameter for quantum simulation.

Adiabatic quantum simulation

Start system polarized along the y direction with B(t=0) large

Turn the laser-ion interaction on and ramp the B field slowly to zero.

The system evolves from a state polarized along the y axis to the ground state of the system in no artificial magnetic field.

The evolution operator in the interaction representation can be written as

$$U(t,0) = e^{-i H_{phon} t} \; T \exp\left[-\frac{i}{\hbar}\int_0^t \left(V_I(t') + H_B(t')\right] dt'\right]$$

$$V_I(t) = -\sum_{j=1}^{N}\sum_{\alpha=1}^{N} \hbar \Omega_j \delta k \sqrt{\frac{\hbar}{2MW_\alpha}}\, b_j^\alpha \left(a_\alpha^{+} e^{iW_\alpha t} + a_\alpha e^{-iW_\alpha t}\right)\sigma_j^x \sin\mu t$$

Note that $[V_I(t), V_I(t')] = -\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\alpha=1}^{N} \frac{\hbar^2 \Omega_j^2 \delta k^2 \hbar}{2mW_\alpha} b_j^\alpha b_i^\alpha \sigma_j^x \sigma_i^x$

$$* \left\{e^{iW_\alpha(t-t')}(-1) + e^{-iW_\alpha(t-t')}(+1)\right\} \sin\mu t \sin\mu t'$$

$$= -\sum_{ij=1}^{N}\sum_{\alpha=1}^{N} \frac{\hbar^3 \Omega_j^2 \delta k^2}{2mW_\alpha} b_i^\alpha b_j^\alpha \left(-2i\sin W_\alpha(t-t')\right)\sigma_i^x \sigma_j^x$$

$$* \sin\mu t \sin\mu t'$$

But this commutator commutes with everything else except for the $H_B(t)$ term.

If we assume $\int_0^t B(t')\,dt' \ll 1$ then any terms coming from ~~integrals of~~ commutators of $H_B(t)$ are small because the term in front is small.

So we proceed as before - define

$$W_I(t) = \int_0^t dt'\, V_I(t')\,dt'$$

and we find

$$U(t,0) = e^{-\frac{i}{\hbar}H_{phonon}t}\, e^{-\frac{i}{\hbar}W_I(t)}\, U_{spin}(t,0) \qquad \frac{1}{2}[W_I(t), V_I(t)]$$

where
$$U_{spin}(t,0) = T\exp\left[-\frac{i}{\hbar}\int_0^t dt'\left\{\sum_{ij} J_{ij}(t)\,\sigma_i^x\sigma_j^x + \sum_i B(t)\,\sigma_j^y\right\}\right]$$

With $J_{ij}(t)$ found from performing some integrals

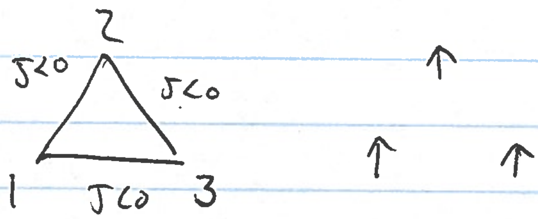
$$J_{ij}(t) = -\frac{\hbar}{2}\sum_{\alpha=1}^{N}\Omega_i\Omega_j\, b_i^\alpha b_j^\alpha \frac{(\delta k)^2\hbar}{2m\omega_\alpha}\frac{1}{\omega_\alpha^2-\mu^2}$$

$$* \left\{ \underset{\substack{\uparrow \\ \text{Constant piece}}}{\omega_\alpha} - \underbrace{\omega_\alpha\cos 2\mu t - 2\mu\sin\omega_\alpha t\sin\mu t}_{\text{time-dependent piece.}} \right\}$$

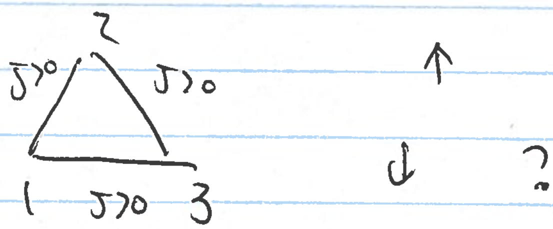
The constant piece looks like an Ising model

$$H_{Ising} = \sum_{ij} J_{ij}\, \sigma_i^x \sigma_j^x$$

On a triangle we can have ferromagnetic states or frustrated states



gs ferromagnetic



gs frustrated.

The full spin problem looks like an Ising model in a transverse field. For the moment let us neglect the time dependent exchange.

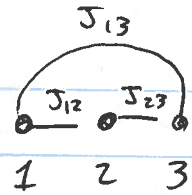
$$H_{transverse} = \sum_{ij} J_{ij}\, \sigma_i^x \sigma_j^x + B(t) \sum_i \sigma_i^y$$

This Hamiltonian has two symmetries on the chain

1.) spin inversion symmetry — If we take $\sigma_x \to -\sigma_x$ $\sigma_z \to -\sigma_z$ then H is invariant as are the spin commutation relations $\Rightarrow$ can classify states as even or odd under change of sign of spin

2.) spatial reflection symmetry - The ion traps are symmetric, reflected about the center - classify states as even/odd under spatial reflection/inversion.

example - 3 site chain



$J_{12} = J_{23} \Rightarrow$ spatial symmetry.

classify states in the x spin basis ($\uparrow$ or $\downarrow$ along x)

spin E, space E $\qquad |\sigma_1^x \, \sigma_2^x \, \sigma_3^x\rangle$

$\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle\right)$

$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle\right)$

$\frac{1}{2}\left(|\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\downarrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle\right)$

O E

$\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle\right)$

$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\downarrow\rangle\right)$

$\frac{1}{2}\left(|\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle - |\downarrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle\right)$

E O

$\frac{1}{2}\left(|\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle + |\downarrow\downarrow\uparrow\rangle - |\uparrow\downarrow\downarrow\rangle\right)$

O O

$\frac{1}{2}\left(|\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle - |\downarrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle\right)$

From this we can compute H in each symmetry sector (block diagonalization). It turns out the ground state is in the odd spin even space sector.

You will explore this on the homework.

~~We can include the~~

~~Three~~ Four diffrent approximations for the time evolution -

1.) Adiabatic spin evolution

a.) Ignore time dependant exchange terms

b.) assume $B(t)$ changes slow enough that the system remains in the grandstate

c.) For each t, find $B(t)$, diagonalize H take the grand state at each t as the eigenstate

2.) Sudden approximation.

same as above except start state along ~~B~~ y direction and turn J and B on together. Project polarized state onto eigenstates to get initial probabilities Then evolve each state adiabatically, fixing the initial probabilities.

3.) Full time dependent spin evolution

evaluate the spin evolution operator by discretizing time and the so-called Trotter formula

$$U(t+\Delta t, 0) = e^{-\frac{i}{\hbar} H(t+\frac{\Delta t}{2})\cdot\Delta t}\, U(t,0)$$

start from $U(0,0) = \mathbb{1}$ to get

$$U(t+\Delta t, 0) = e^{-\frac{i}{\hbar} H(t+\frac{\Delta t}{2})\cdot\Delta t}\, e^{-\frac{i}{\hbar} H(t-\frac{\Delta t}{2})\Delta t} \cdots e^{-\frac{i}{\hbar} H(\frac{\Delta t}{2})\Delta t}\, \mathbb{1}$$

and evolve states forward in time

4.) Full evolution including phonons - evolve the wavefunction including phonons from its initial state.

Surprisingly, over a wide range of parameters 4 and 2 agree very well and agree with experiment.