It turns out that there are five critically important operator identities that we need to know to do just about anything in quantum mechanics. They are the Leibnitz (or product rule) identity, the Hadamard lemma (and braiding identity), the exponential reordering identity, the Baker-Campbell-Hausdorff identity and the exponential disentangling identity.

All but one of them are quite elementary to prove. One is hard. Really hard. So we only do it for special cases.

1.) The Leibnitz or product rule

$$[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A},\hat{C}] + [\hat{A},\hat{B}]\hat{C}$$

proof: $[\hat{A}, \hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A}$

add 0 $= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A}$

regroup $= (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A})$

$= [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$

It acts just like a differentiation product rule, where the single operator in the commutator is like a derivative and the product of operators is like a product of functions.

It has other forms too, like $[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$.

Related to this is the Jacobi identity

$$[\hat{A},[\hat{B},\hat{C}]] + [\hat{B},[\hat{C},\hat{A}]] + [\hat{C},[\hat{A},\hat{B}]] = 0$$

which I don't find I use very much. This can be proven just by writing it out

$$\hat{A}[\hat{B},\hat{C}] - [\hat{B},\hat{C}]\hat{A} + \hat{B}[\hat{C},\hat{A}] - [\hat{C},\hat{A}]\hat{B} + \hat{C}[\hat{A},\hat{B}] - [\hat{A},\hat{B}]\hat{C}$$

$$= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} - \hat{B}\hat{C}\hat{A} + \hat{C}\hat{B}\hat{A} + \hat{B}\hat{C}\hat{A} - \hat{B}\hat{A}\hat{C}$$

$$- \hat{C}\hat{A}\hat{B} + \hat{A}\hat{C}\hat{B} + \hat{C}\hat{A}\hat{B} - \hat{C}\hat{B}\hat{A} - \hat{A}\hat{B}\hat{C} + \hat{B}\hat{A}\hat{C}$$

$$= 0$$

Next up is the Hadamard Lemma!

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots$$
$$= \sum_{n=0}^{\infty} \underbrace{[\dots[\hat{A},\hat{B}]\dots]_n}_{\text{n fold nested commutator}} \frac{1}{n!}$$

Proof: Consider $f(x) = e^{x\hat{A}}\hat{B}e^{-x\hat{A}}$

$$f(0) = \hat{B}$$
$$f'(0) = \hat{A}e^{x\hat{A}}\hat{B}e^{-x\hat{A}} - e^{x\hat{A}}\hat{B}e^{-x\hat{A}}\hat{A}\Big|_{x=0}$$
$$= [\hat{A},\hat{B}]$$
$$f''(0) = \hat{A}f'(0) - f'(0)\hat{A}$$
$$= [\hat{A},[\hat{A},\hat{B}]]$$

In general, if we assume

$$f^{(n)}(0) = [\hat{A},\dots[\hat{A},B]\dots]_n$$

then

$$f^{(n+1)}(0) = \hat{A}f^{(n)}(0) - f^{(n)}(0)\hat{A}$$
$$= [\hat{A},\dots[\hat{A},\hat{B}]\dots]_{n+1}$$

Then using Taylor's theorem with $x=1$ gives us the proof.

$$\boxed{e^{\hat{A}}\hat{B}e^{-\hat{A}} = \sum_{n=0}^{\infty}\frac{1}{n!}[\hat{A},[\dots[\hat{A},\hat{B}]\dots]_n}$$

The braiding identity is even more powerful. It comes from the fact that

$$e^{\hat{A}}\hat{B}^n e^{-\hat{A}} = \underbrace{e^{\hat{A}}\hat{B}e^{-\hat{A}}\,e^{\hat{A}}\hat{B}e^{-\hat{A}}\dots e^{\hat{A}}\hat{B}e^{-\hat{A}}}_{n\text{ triples}}$$
$$= (e^{\hat{A}}\hat{B}e^{-\hat{A}})^n$$

Hence, any operator series (or any function that can be expressed as a series) satisfies

$$f(\hat{B}) = \sum_n c_n \hat{B}^n$$

then

$$e^{\hat{A}}f(\hat{B})e^{-\hat{A}} = \sum_n c_n e^{\hat{A}}\hat{B}^n e^{-\hat{A}} = \sum_n c_n (e^{\hat{A}}\hat{B}e^{-\hat{A}})^n$$
$$= f(e^{\hat{A}}\hat{B}e^{-\hat{A}})$$

So

$$\boxed{\begin{aligned} e^{\hat{A}}f(\hat{B})e^{-\hat{A}} &= f(e^{\hat{A}}\hat{B}e^{-\hat{A}}) \\ &= f\Big(\sum_{n=0}^{\infty}\frac{1}{n!}[\hat{A},\dots[\hat{A},\hat{B}]\dots]_n\Big)\end{aligned}}$$

This is the braiding relation.

The exponential reordering identity is most powerful in simple cases as we will see.

If I have $e^{\hat{A}} e^{\hat{B}}$ and I want to interchange the places what do I get

In other words, $e^{\hat{A}} e^{\hat{B}} =$ what $e^{\hat{A}}$?

The way to solve this is with braiding:

$$e^{\hat{A}} e^{\hat{B}} e^{-\hat{A}} = e^{(e^{\hat{A}}\hat{B}e^{-\hat{A}})}$$
$$= e^{\hat{B} + [\hat{A},\hat{B}] + \frac{1}{2}[\hat{A},[\hat{A},\hat{B}]] + \cdots}$$

multiply both sides by $e^{\hat{A}}$ on the right

$$\boxed{e^{\hat{A}} e^{\hat{B}} = e^{\hat{B} + [\hat{A},\hat{B}] + \frac{1}{2}[\hat{A},[\hat{A},\hat{B}]] + \cdots} e^{\hat{A}}}$$

In general, this is a complicated mess, but what if $[\hat{A},[\hat{A},\hat{B}]] = 0$, like what happens if $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$?

Then, we get
$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{B} + [\hat{A},\hat{B}]} e^{\hat{A}}$$

But if $[\hat{B},[\hat{A},\hat{B}]] = 0$ too (as it would in our example) then we have

$$\boxed{\begin{array}{l} e^{\hat{A}} e^{\hat{B}} = e^{[\hat{A},\hat{B}]} e^{\hat{B}} e^{\hat{A}} \\ \text{when } [\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0 \end{array}}$$

This is a very powerful identity that comes up often.

We are now ready for the hard identity, Baker-Campbell-Hausdorff. It asks the question

$$e^{\hat{A}} e^{\hat{B}} = e^{\text{what?}}$$

which is kind of half way to the exponential re-ordering identity. But it is much harder to work out. In fact, it leads to an infinite series of commutators, just like the Hadamard lemma, but the coefficients have no regular structure, even though one can find equations that determine them.

As a warmup, lets work out the case when $[\hat{A},\hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$.
This is called the Weyl form of the identity.

Then define

$$f(\lambda) = e^{-\lambda\hat{A}}\, e^{\lambda(\hat{A}+\hat{B})}\, e^{-\lambda\hat{B}}$$

$$\frac{df(\lambda)}{d\lambda} = -\hat{A}e^{-\lambda\hat{A}}\, e^{\lambda(\hat{A}+\hat{B})}\, e^{-\lambda\hat{B}} + e^{-\lambda\hat{A}}(\hat{A}+\hat{B})e^{\lambda(\hat{A}+\hat{B})}e^{-\lambda\hat{B}} - e^{-\lambda\hat{A}}\, e^{\lambda(\hat{A}+\hat{B})}\, e^{-\lambda\hat{B}}\,\hat{B}$$

But $\hat{A}e^{-\lambda\hat{A}} = e^{-\lambda\hat{A}}\hat{A}$ and $\hat{B}e^{\lambda(\hat{A}+\hat{B})}$ can be rewritten with Hadamard

$$e^{\lambda(\hat{A}+\hat{B})}\, e^{-\lambda(\hat{A}+\hat{B})}\,\hat{B}\, e^{\lambda(\hat{A}+\hat{B})}$$

$$= e^{\lambda(\hat{A}+\hat{B})}\left(\hat{B} - \lambda[\hat{A}+\hat{B},\hat{B}] + \frac{\lambda^2}{2}[\hat{A}+\hat{B},[\hat{A}+\hat{B},\hat{B}]] \ldots\right)$$

$$= e^{\lambda(\hat{A}+\hat{B})}\left(\hat{B} - \lambda[\hat{A},\hat{B}] + 0\right)$$

So $\frac{df(\lambda)}{d\lambda} = -\lambda[\hat{A},\hat{B}]f(\lambda) \Rightarrow f(\lambda) = e^{-\frac{\lambda^2}{2}[\hat{A},\hat{B}]}f(0)$

or $\quad e^{-\frac{\lambda^2}{2}[\hat{A},\hat{B}]} = e^{-\lambda\hat{A}}\, e^{\lambda(\hat{A}+\hat{B})}\, e^{-\lambda\hat{B}}$

multiply on left with $e^{\lambda\hat{A}}$ on right with $e^{\lambda\hat{B}}$ to get

$$e^{\lambda\hat{A}}\, \underset{\substack{\uparrow \\ \text{commutes with everything}}}{e^{-\frac{\lambda^2}{2}[\hat{A},\hat{B}]}}\, e^{\lambda\hat{B}} = e^{\lambda(\hat{A}+\hat{B})}$$

So $\quad e^{\lambda\hat{A}}\, e^{\lambda\hat{B}} = e^{\lambda(\hat{A}+\hat{B}) + \frac{\lambda^2}{2}[\hat{A},\hat{B}]}$

now set $\lambda = 1$

$$\boxed{\begin{array}{l} e^{\hat{A}}\, e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]} \\ \text{when } [\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0 \end{array}}$$

This is the easy case of BCH and the version most people know. Now we work out the general case!

We need another identity!

$$\frac{d}{dx}e^{\hat{A}(x)} = \int_0^1 dy\, e^{(1-y)\hat{A}(x)}\frac{d\hat{A}(x)}{dx}e^{y\hat{A}(x)}$$

This looks wierd, but it takes this form because

$$\left[\hat{A}(x), \frac{d\hat{A}}{dx}\right] \neq 0 \quad \text{in general.}$$

But $\quad \frac{d}{dx}e^{\hat{A}(x)} = \frac{d}{dx}\left[1 + \hat{A} + \frac{1}{2}\hat{A}^2 + \frac{1}{6}\hat{A}^3 + \ldots\right]$

$$= \hat{A}' + \frac{1}{2}(\hat{A}'\hat{A} + \hat{A}\hat{A}') + \frac{1}{6}(\hat{A}'\hat{A}^2 + \hat{A}\hat{A}'\hat{A} + \hat{A}^2\hat{A}')\ldots$$

The n'th order terms in the series will have m powers of $\hat{A}$, $\hat{A}'$ and $n-m-1$ powers of $\hat{A}$; they are divided by $n!$. We can rewrite this as

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{(n+m+1)!}\hat{A}^n \hat{A}' \hat{A}^m$$

Now expand each term in the integrand

$$\int_0^1 dy \sum_{n=0}^{\infty} \frac{(1-y)^n \hat{A}^n}{n!}\hat{A}' \sum_{m=0}^{\infty}\frac{y^m}{m!}\hat{A}^m$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\hat{A}^n \hat{A}' \hat{A}^m}{n!\,m!} \int_0^1 dy\, \underset{\substack{\uparrow \\ \text{Beta function}}}{(1-y)^n y^m}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\hat{A}^n \hat{A}' \hat{A}^m}{n!\,m!}\cdot \frac{n!\,m!}{(n+m+1)!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\hat{A}^n \hat{A}' \hat{A}^m}{(n+m+1)!}$$

So $\quad \frac{d}{dx}e^{\hat{A}} = \int_0^1 dy\, e^{(1-y)\hat{A}}\hat{A}' e^{y\hat{A}}$

multiply on left by $e^{-\hat{A}}$

$$e^{-\hat{A}}\frac{d}{dx}e^{\hat{A}} = \int_0^1 dy\, \underbrace{e^{-y\hat{A}}\hat{A}' e^{y\hat{A}}}_{\text{Hadamard}}$$

$$= \int_0^1 dy\left(\hat{A}' - y[\hat{A},\hat{A}'] + \frac{y^2}{2}[\hat{A},[\hat{A},\hat{A}']] + \cdots\right)$$

$$= \hat{A}' + \frac{1}{2}[\hat{A}',\hat{A}] + \frac{1}{6}[[\hat{A}',\hat{A}],\hat{A}] + \cdots$$

To get BCH, we define

$$e^{x\hat{A}}e^{x\hat{B}} = e^{\hat{G}(x)} = e^{x\hat{G}_1 + x^2\hat{G}_2 + x^3\hat{G}_3 + \cdots}$$

$$e^{-x\hat{B}}e^{-x\hat{A}}\frac{d}{dx}\left(e^{x\hat{A}}e^{x\hat{B}}\right) = e^{-\hat{G}(x)}\frac{d}{dx}e^{\hat{G}(x)}$$

The LHS can be evaluated directly $\quad$ [Note $e^{-x\hat{B}}e^{-x\hat{A}} = e^{-\hat{G}(x)}$]

$$e^{-x\hat{B}}e^{-x\hat{A}}\frac{d}{dx}\left(e^{x\hat{A}}e^{x\hat{B}}\right)$$

$$= e^{-x\hat{B}}e^{-x\hat{A}}\left(\hat{A}e^{x\hat{A}}e^{x\hat{B}} + e^{x\hat{A}}e^{x\hat{B}}\hat{B}\right)$$

$$= \hat{B} + e^{-x\hat{B}}\hat{A}e^{x\hat{B}}$$

$$= \hat{B} + \hat{A} - x[\hat{B},\hat{A}] + \frac{x^2}{2}[\hat{B},[\hat{B},\hat{A}]] + \cdots$$

But we also have $\quad e^{-\hat{G}(x)}\frac{d}{dx}e^{\hat{G}(x)} = G' + \frac{1}{2}[G',G] + \frac{1}{6}[[G',G],G] + \cdots$

Use the power series for $\hat{G}(x) = x\hat{G}_1 + x^2\hat{G}_2 + x^3\hat{G}_3 + \cdots$

$$G' = \hat{G}_1 + 2x\hat{G}_2 + 3x^2\hat{G}_3 + \cdots$$

$$\frac{1}{2}[G',G] = \frac{1}{2}[\hat{G}_1 + 2x\hat{G}_2 + 3x^2\hat{G}_3 + \cdots,\ x\hat{G}_1 + x^2\hat{G}_2 + x^3\hat{G}_3 \cdots]$$

$$= \frac{1}{2}\Big(x^2[\hat{G}_1,\hat{G}_2] + 2x^2[\hat{G}_2,\hat{G}_1] + x^3[\hat{G}_1,\hat{G}_3] + 3x^3[\hat{G}_3,\hat{G}_1]\Big) + \cdots$$

$$= \frac{1}{2}x^2[\hat{G}_2,\hat{G}_1] + x^3[\hat{G}_3,\hat{G}_1] + \cdots$$

$$\frac{1}{6}[[G',G],G] = \frac{1}{6}[[\hat{G}_1, x^2\hat{G}_2], x\hat{G}_1] = \frac{x^3}{6}[[\hat{G}_1,\hat{G}_2],\hat{G}_1]$$

So we have (6)

$$e^{-\hat{G}(x)} \frac{d}{dx} e^{\hat{G}(x)} = \hat{G}_1 + 2x\hat{G}_2 + 3x^2 G_3 + \frac{x^2}{2}[\hat{G}_2, \hat{G}_1] + O(x^3)$$

but this equals

$$\hat{A} + \hat{B} + x[\hat{A}, \hat{B}] + \frac{x^2}{2}[\hat{B}, [\hat{B}, \hat{A}]] + \cdots$$

So

$$G_1 = \hat{A} + \hat{B}$$

$$G_2 = \frac{1}{2}[\hat{A}, \hat{B}]$$

$$3G_3 + \frac{1}{2}[G_2, G_1] = \frac{1}{2}[\hat{B}, [\hat{B}, \hat{A}]]$$

$$G_3 = \frac{1}{6}[\hat{B}, [\hat{B}, \hat{A}]] - \frac{1}{6}\left[\frac{1}{2}[\hat{A}, \hat{B}], \hat{A} + \hat{B}\right]$$

$$= \frac{1}{6}[\hat{B}, [\hat{B}, \hat{A}]] + \frac{1}{12}[\hat{A}, [\hat{A}, \hat{B}]] - \frac{1}{12}[\hat{B}, [\hat{B}, \hat{A}]]$$

$$G_3 = \frac{1}{12}[\hat{B}, [\hat{B}, \hat{A}]] + \frac{1}{12}[\hat{A}, [\hat{A}, \hat{B}]]$$

But if we set $x = 1$, then

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{G}(x=1)} = e^{\hat{G}_1 + \hat{G}_2 + \hat{G}_3 + \cdots}$$

So $$\boxed{e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \frac{1}{12}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{12}[\hat{B}, [\hat{B}, \hat{A}]] + \cdots}}$$

which is the BCH. We can work out higher order terms but it is painful.

The fifth identity is exponential disentangling and it uses a theorem from Lie algebras:
Any identity expressed in terms of exponentials of the generators of a Lie algebra that is proved for one faithful representation holds in all representations.

So, the identity we worked out last time

$$\exp\left[i\, 2\frac{\vec{v} \cdot \hat{\vec{S}}}{\hbar}\right] = \exp\left[\alpha \frac{2\hat{S}_+}{\hbar}\right] \exp\left[\beta \frac{2\hat{S}_z}{\hbar}\right] \exp\left[\gamma \frac{2\hat{S}_-}{\hbar}\right]$$

with α, β, and γ determined as before, holds for any angular momentum representation!

This is amazingly powerful. Prove something about Pauli spin matrices and it holds for all j!! We will use this again later.