Lecture 34 What is a photon?

In the next two lectures we will summarize quantum optics with the goals of (1) establishing precisely what a photon is and (2) describing how quantum optics principles are employed in LIGO to improve the precision of the measurements. It will be a "crash course." You need to review ladder operators for SHO's and coherent states for this lecture.

We begin with a classical description of an electric field given by

$$\vec{E}(r,t) = \sum_{l} \vec{\varepsilon}_l\, E_l(t)\, e^{i\vec{k}_l\cdot\vec{r}} + c.c.$$

where we are describing a real traveling wave. Here l denotes the mode, described by the wavevector $\vec{k}_l$ and polarization $\vec{\varepsilon}_l$. Here, we will focus on linear polarization only. The function $E_l(t)$ can be complex valued.

We require the wave to satisfy maxwell's equations

$$\nabla\cdot\vec{E}(r,t)=0 \qquad \nabla\times\vec{E}(r,t) = -\frac{\partial\vec{B}(r,t)}{\partial t} \qquad \nabla\cdot\vec{B}(r,t)=0 \qquad \nabla\times\vec{B}(r,t) = \frac{1}{c^2}\frac{\partial\vec{E}(r,t)}{\partial t}$$

Using the fact that $\nabla e^{i\vec{k}_l\cdot\vec{r}} = i\vec{k}_l\, e^{i\vec{k}_l\cdot\vec{r}}$, $\nabla\cdot E=0 \Rightarrow \vec{k}_l$ is $\perp$ to $\vec{\varepsilon}_l$, $\nabla\times E = -\frac{\partial B}{\partial t} \Rightarrow \vec{B}(r,t) \propto \vec{k}_l\times\vec{\varepsilon}_l$ then $\nabla\cdot B=0$ is automatically satisfied. We end up with $\frac{\partial^2 E_l(t)}{\partial t^2} = -\frac{k_l^2}{c^2}E_l(t)$. We define $\omega_l = \frac{k_l}{c}$ = angular frequency. Then $E_l(t) = E_l(0)\, e^{-i\omega_l t}$

$$\text{Then}\quad \vec{B}(r,t) = \sum_l \frac{\vec{k}_l\times\vec{\varepsilon}_l}{\omega_l}\, E_l(t)\, e^{i\vec{k}_l\cdot\vec{r}} + c.c.$$

We define the single photon amplitude as $\mathcal{E}^{(1)}_l = \sqrt{\frac{\hbar\omega_l}{2\varepsilon_0 V_l}}$

and introduce V_l as the quantization volume for mode l.
We will be working in a volume L^3 with peridic boundary conditions so $\vec{k}_l = \frac{2\pi}{L}(n_x, n_y, n_z)$ $n_\alpha \in \mathbb{Z}$. The polarization is one of two directions perpendicular to k_l.
Note that we use $\vec{k}_l$, $\vec{\varepsilon}_l$ and $\vec{k}_l\times\vec{\varepsilon}_l$ as a triad to define the coordinates of the 3 dimensions. We also choose $\vec{\varepsilon}_{-l} = \vec{\varepsilon}_l$ when $l: \frac{2\pi}{L}(n_x, n_y, n_z; \vec{\varepsilon}_l)$ and $-l: \frac{2\pi}{L}(-n_x, -n_y, -n_z; \vec{\varepsilon}_{-l})$
This will help us with calculating the total energy.

But before that, we note two more things: We will be writing $E_l(t)$ as $i\,\mathcal{E}^{(1)}_l\,\alpha_l(t)$ with $\alpha_l(t)$ possibly complex.
Then we also define the two quadrature parameters

$$Q_l(t) = \sqrt{\frac{\hbar}{2}}\left(\alpha_l(t) + \alpha_l^*(t)\right) \qquad P_l(t) = \sqrt{\frac{\hbar}{2}}\,\frac{\left(\alpha_l(t) - \alpha_l^*(t)\right)}{i}$$

The energy is given by $E = \frac{\varepsilon_0}{2}\int d^3r\,(\vec{E}^2 + c^2\vec{B}^2)$

In calculating this, we note that

$$\int d^3r\, e^{i\vec{k}_\ell\cdot\vec{r} - \vec{k}_{\ell'}\cdot\vec{r}} = \delta_{\vec{k}_\ell,\vec{k}_{\ell'}} V$$

we end up with two types of terms - those where $\ell=\ell'$ and those with $\ell=-\ell'$

Using $\vec{\varepsilon}_\ell = \vec{\varepsilon}_{-\ell}$ then shows (you should do this)

$$E = \frac{\varepsilon_0}{2} V \sum_\ell \{ 2|E_\ell(t)|^2 - E_\ell(t)E_{-\ell}(t) - E_\ell^*(t)E_{-\ell}^*(t) + 2|E_\ell(t)|^2 + E_\ell(t)E_{-\ell}(t) + E_\ell^*(t)E_{-\ell}^*(t) \}$$

$$= 2\varepsilon_0 V \sum_\ell (\mathcal{E}_\ell^{(1)})^2 |\alpha_\ell|^2 = \sum_\ell \hbar\omega_\ell |\alpha_\ell|^2$$

Now we quantize! Let $Q_\ell \to \hat{Q}_\ell$ $\quad P_\ell \to \hat{P}_\ell$ $\quad [\hat{Q}_\ell, \hat{P}_{\ell'}] = i\hbar\delta_{\ell\ell'}$

Then we define $\quad \hat{a}_\ell = \frac{1}{\sqrt{2\hbar}}(\hat{Q}_\ell + i\hat{P}_\ell) \quad \hat{a}_\ell^\dagger = \frac{1}{\sqrt{2\hbar}}(\hat{Q}_\ell - i\hat{P}_\ell)$

Written in terms of quadratures $H = \sum_\ell \frac{\omega_\ell}{2}(\hat{Q}_\ell^2 + \hat{P}_\ell^2)$ which shows the relationship with the simple harmonic oscillator.

Back to the field, we now have an operator

$$\hat{\vec{E}}(\vec{r}) = \hat{E}^{(+)} + \hat{E}^{(-)} = i\sum_\ell \vec{\varepsilon}_\ell\, \mathcal{E}_\ell^{(1)}\, \hat{a}_\ell\, e^{i\vec{k}_\ell\cdot\vec{r}} + \text{h.c.}$$

The eigenstates are labeled by number operator eigenstates

$$|n_1, n_2, n_3, \ldots\rangle = \frac{(\hat{a}_1^\dagger)^{n_1}}{\sqrt{n_1!}}\frac{(\hat{a}_2^\dagger)^{n_2}}{\sqrt{n_2!}}\cdots|0\rangle$$

and $E_{n_1\ldots} = \sum_\ell \hbar\omega_\ell(n_\ell + \frac{1}{2})$. Usually, we only include nonzero n_ℓ's in the labeling. Note the vacuum has "infinite energy" given by $\sum_\ell \hbar\omega_\ell \frac{1}{2}$. But we avoid this energy by focusing on the excitation energy given by $E^{ex}_{n_1\ldots n_n} = \sum_\ell \hbar\omega_\ell n_\ell$.

We will be working in the Heisenberg representation where $\hat{a}_\ell(t) = \hat{a}_\ell e^{-i\omega_\ell t}$ and $\hat{a}_\ell^\dagger(t) = \hat{a}_\ell^\dagger e^{i\omega_\ell t}$.

Now consider a single mode state $|n_\ell\rangle$. The average of the electric field vanishes

$$\langle n_\ell|\hat{\vec{E}}(\vec{r},t)|n_\ell\rangle = \langle n_\ell| i\,\mathcal{E}_\ell^{(1)}\left(\hat{a}_\ell e^{i\vec{k}_\ell\cdot\vec{r} - i\omega_\ell t} - \hat{a}_\ell^\dagger e^{-i\vec{k}_\ell\cdot\vec{r} + i\omega_\ell t}\right)|n_\ell\rangle$$

$= 0$ since the $\hat{a}_\ell^\dagger$ and $\hat{a}_\ell$ operators are unbalanced.

We calculate the fluctuations from $\langle n_\ell|\hat{\vec{E}}\cdot\hat{\vec{E}}|n_\ell\rangle$

$$= -\vec{\varepsilon}_\ell\cdot\vec{\varepsilon}_\ell\,(\mathcal{E}_\ell^{(1)})^2 \langle n_\ell|\hat{a}_\ell^2 e^{i(2\vec{k}_\ell\cdot\vec{r} - 2\omega_\ell t)} - \hat{a}_\ell\hat{a}_\ell^\dagger - \hat{a}_\ell^\dagger\hat{a}_\ell + \hat{a}_\ell^{\dagger 2} e^{-i(2\vec{k}_\ell\cdot\vec{r} - 2\omega_\ell t)}|n_\ell\rangle$$

$= (\mathcal{E}_\ell^{(1)})^2(2n_\ell + 1)$ $\quad$ so $\quad \Delta\vec{E} = \mathcal{E}_\ell^{(1)}\sqrt{2n_\ell + 1}$.

In particular, the vaccum $(n_\ell = 0)$ has fluctuations. This is real and responsible for spontaneous emission, Lamb shift, g-2, Casimir effect, etc.

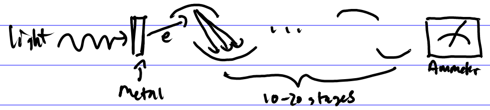
A similar calculation shows that

$$\langle 0|\hat{P}_\ell|0\rangle = \langle 0|\hat{Q}_\ell|0\rangle = 0$$

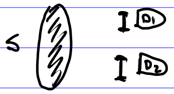
$$\langle 0|\hat{P}_\ell^2|0\rangle = \langle 0|\hat{Q}_\ell^2|0\rangle = \sqrt{\frac{\hbar}{2}}$$

So $\Delta Q_\ell \, \Delta P_\ell = \frac{\hbar}{2} \Rightarrow$ minimum uncertainty state. (same as SHO ground state).

Our next step is to describe photo detection. We will describe a photo multiplier tube which uses the photo electric effect and a cascade



A single photon releases an electron which is accelerated and leads to huge amplification of electrons which can be measured as an electron pulse. The important aspects are these detectors are efficient and fast (there are more efficient fast detectors, but we focus on these which is all we need).



Suppose light is traveling with a areal profile of S and impinging on detectors 1 and 2

The probability to detect one photon is $dP(\vec{r},t) = W^{(1)}(\vec{r},t)\, dS\, dt$

with $W^{(1)}(\vec{r},t) = s \, \|\hat{E}^+(\vec{r},t)|\psi\rangle\|^2$ = single photon detection prob.

s = sensitivity of the detector, $\hat{E}^+$ is the $-i\omega t$ term in $\hat{E}$ and $|\psi\rangle$ is the photon state.

The probability to detect two photons is $dP(\vec{r}_1 t_1, \vec{r}_2 t_2) = W^{(2)}(\vec{r}_1 t_1, \vec{r}_2 t_2)\, dS\, dt$

with $W^{(2)}(\vec{r}_1 t_1, \vec{r}_2 t_2) = s^2 \, \|\hat{E}^+(\vec{r}_2 t_2)\hat{E}^+(\vec{r}_1 t_1)|\psi\rangle\|^2$

$$\vec{E}^+_\ell(\vec{r},t) = i\, \vec{\mathcal{E}}_\ell\, \varepsilon^{(1)}_\ell\, \hat{a}_\ell\, e^{i(\vec{k}_\ell\cdot\vec{r} - i\omega_\ell t)}$$

Consider a single-photon state. We have $|1_\ell\rangle = \hat{a}^+_\ell|0\rangle$.

Then $W^{(1)}(\vec{r},t) = s\, \| i\, \vec{\mathcal{E}}_\ell\, \varepsilon^{(1)}_\ell\, e^{i(\vec{k}_\ell\cdot\vec{r} - i\omega_\ell t)}\, \hat{a}_\ell \hat{a}^+_\ell |0\rangle \|^2$

but $\hat{a}_\ell \hat{a}^+_\ell |0\rangle = [\hat{a}_\ell, \hat{a}^+_\ell]|0\rangle = |0\rangle$

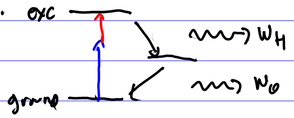
so $W^{(1)}(\vec{r},t) = s$

$$W^{(2)}(\vec{r}_1,t_1,\vec{r}_2,t_2) = s^2 \| i\, \vec{\mathcal{E}}_\ell\, \varepsilon^{(1)}_\ell\, e^{i(\vec{k}_\ell\cdot\vec{r}_1 - \omega_\ell t_1)}\; i\, \vec{\mathcal{E}}_\ell\, \varepsilon^{(1)}_\ell\, e^{i(\vec{k}_\ell\cdot\vec{r}_2 - i\omega_\ell t_2)}\, \hat{a}_\ell \hat{a}_\ell \hat{a}^+_\ell |0\rangle\|^2$$

$= 0$ since $\hat{a}_\ell \hat{a}_\ell \hat{a}^+_\ell |0\rangle = \hat{a}_\ell |0\rangle = 0$

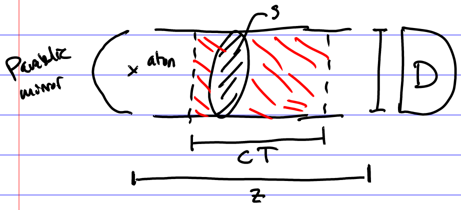
So a single photon state can only be measured once! Measuring one photon alters the quantum state so it cannot be measured again.

We now describe how one makes a single-photon source.



Two photon excitation process, and a two photon decay. The two photons emerge within a few ns

So, by observing ω_H, one is sure a photon ω_0 is emitted within the next 10-15 ns.



Collect the atoms at the focus of a parabolic mirror. Then the photon will "live" in a volume given by ScT with T the lifetime of the excited state,

Recall $\frac{dP^{(1)}}{dt\,dS} = s\,|\mathcal{E}_\ell^{(1)}|^2 = \frac{s\hbar\omega_\ell}{2\varepsilon_0 V_\ell} = \frac{s\hbar\omega_\ell}{2\varepsilon_0 ScT}$

So $\frac{dP^{(1)}}{dt} = \int dS\,\frac{dP^{(1)}}{dt\,dS} = \frac{s\hbar\omega_\ell}{2\varepsilon_0 cT_\ell}$ and $\int_T \frac{dP^{(1)}}{dt} = \frac{s\hbar\omega_\ell}{2\varepsilon_0 c} = 1$ ← for a perfect detector

So $S_{perfect}$ = perfect efficiency = $\frac{2\varepsilon_0 c}{\hbar\omega_\ell}$

We define quantum efficiency as $s = \eta\, S_{perfect}$ then $\boxed{w^{(1)} = \frac{\eta}{ST}}$

The real photon emitted is in a wave packet

$$|\psi\rangle = \sum_\ell c_\ell\,|1_\ell\rangle$$

with $c_\ell = \frac{K\,e^{i\omega t_0}}{\omega_\ell - \omega_0 + i\Gamma/2}$ = Lorentzian lineshape

K = normalization constant $= \sqrt{\frac{c\Gamma}{L}}$ ω_0 = frequency of excited state Γ = lifetime

Assume the photon is emitted at time t_0. It must travel a distance z to reach the detector. So

$$w^{(1)}(z,t) = s\,\|\hat{E}^+(z,t)\,|\psi(t_0)\rangle\|^2$$

$$= s\,\Big\|\sum_\ell i\,\vec{\varepsilon}_\ell\,\mathcal{E}_\ell^{(1)}\,\hat{a}_\ell\,e^{i(k_\ell z - \omega_\ell t)}\,|\psi(t_0)\rangle\Big\|^2$$

$$= s\,\Big\|\sum_\ell \vec{\varepsilon}_\ell\,\mathcal{E}_\ell^{(1)}\,\hat{a}_\ell\,e^{-i\omega_\ell(-\frac{z}{c}+t)}\sum_{\ell'}\frac{\sqrt{\frac{c\Gamma}{L}}\,e^{i\omega_{\ell'}t_0}}{(\omega_{\ell'}-\omega_0)+i\Gamma/2}\,\hat{a}^\dagger_{\ell'}|0\rangle$$

but $\hat{a}_\ell\hat{a}^\dagger_{\ell'}|0\rangle = \delta_{\ell\ell'}|0\rangle$ so

$$= s\,\Big\|\sum_\ell i\,\vec{\varepsilon}_\ell\,\mathcal{E}_\ell^{(1)}\sqrt{\frac{c\Gamma}{L}}\,\frac{e^{-i\omega_\ell(t-t_0-\frac{z}{c})}}{(\omega_\ell-\omega_0)+i\Gamma/2}\,|0\rangle\Big\|^2$$

assume $\mathcal{E}_\ell^{(1)} \sim \sqrt{\frac{\hbar\omega_0}{2\varepsilon_0 V}}$ since Γ is small sum over $\ell \to i\int_{-\infty}^{+\infty} d\omega_\ell\,\frac{L}{2\pi c}\left(\frac{c\Gamma}{L}\right)^{1/2}\frac{e^{-i\omega_\ell(t-t_0-\frac{z}{c})}}{\omega_\ell-\omega_0+i\Gamma/2}\,|0\rangle$

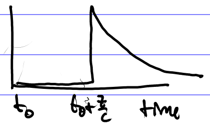
The integral can be done and it gives (contour integral)

$$= \Gamma \frac{L}{2\pi c} \left(\frac{c\pi}{L}\right)^{1/2} e^{-i\omega_0(t-t_0-\frac{z}{c})} \left(-2\pi i\, \Theta(t-t_0-\tfrac{z}{c})\right) e^{-\frac{\Gamma}{2}(t-t_0-\frac{z}{c})} |0\rangle$$

and $w^{(1)}(\mathbf{r},t) = s \frac{\hbar\omega_0}{2\varepsilon_0 V} \frac{L\Gamma}{c} e^{-\Gamma(t-t_0-\frac{z}{c})} \Theta(t-t_0-\frac{z}{c})$

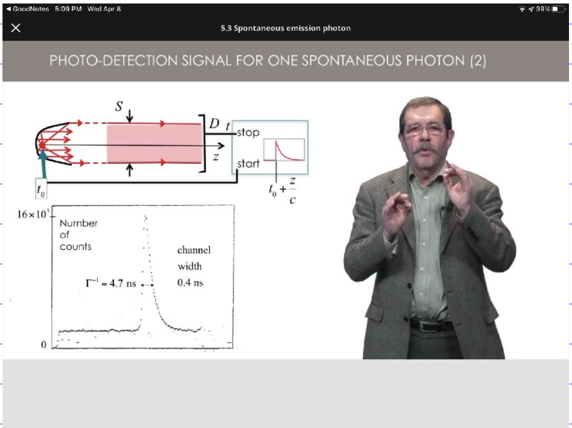
Recall $V = SL$ $\quad s = \frac{2\varepsilon_0 S}{\hbar\omega_0}$ $\quad w^{(1)}(z,t) = \eta \frac{\Gamma}{c} e^{-\Gamma(t-t_0-\frac{z}{c})}\Theta(t-t_0-\frac{z}{c})$

Prob

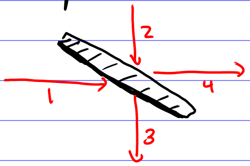


Probability exponentially decays away from initial time.

This has been measured experimentally!



These single photon sources can be verified by measuring them on a beam splitter. We analyze this next



The beamsplitter is a partially silvered mirror that reflects the amplitude with strength r (or $-r$) depending on which side (silvered or not) and transmits with strength t. We have

$$\hat E_3 = r\hat E_1 + t\hat E_2 \qquad \hat E_4 = t\hat E_1 - r\hat E_2$$

The probability in 3 and 4 for single-photon detection is

$$s\,\| E_3^{(+)}(r_3,t_3)|\psi_{in}\rangle\|^2 \quad \text{and} \quad s\,\|E_4^{(+)}(r_4,t_4)|\psi_{in}\rangle\|^2$$

The input state $|\psi_{in}\rangle = (\gamma|1\rangle_1 + \sqrt{1-\gamma^2}|0\rangle_1)\otimes|0\rangle_2$ where the efficiency to detect the heralded photon is γ (not all photons are detected). One finds

$$\frac{dP_3}{dt}(t) = \eta_3 |\gamma|^2 |r|^2 \Theta(t_3-t_0-\tfrac{z_3}{c})\, e^{-\Gamma(t_3-t_0-\frac{z_3}{c})}$$

$$\frac{dP_4}{dt}(t) = \eta_4|\gamma|^2|t|^2\Theta(t_4-t_0-\tfrac{z_4}{c})\, e^{-\Gamma(t_4-t_0-\frac{z_4}{c})}$$

Integrating over a few $\frac{1}{\Gamma}$'s yields

$$N_3 = \eta_3 |\gamma|^2 |r|^2 \qquad N_4 = \eta_4 |\gamma|^2 |t|^2$$

coincidences are of course zero

But in a real experiment, we see coincidences due to dark background current and more than one atom spontaneously emitting in the measurement window.

We define $P_3 = \frac{N_3}{N_H}$ $\quad P_4 = \frac{N_4}{N_H}$ $\quad P_c = \frac{N_c}{N_H}$ $\quad N_H = \#$ heralded

and $\alpha = \frac{P_c}{P_3 P_4} = \frac{N_c N_H}{N_3 N_4}$ For a quantum system, we have $\alpha < 1$. A classical system must have $\alpha \geq 1$, since

$$P_c = \langle W^{(2)} \rangle \geq \langle W^{(1)} \rangle^2 = P_3 P_4$$

we can characterize the single-photon quantum nature by observing $\alpha < 1$

$\nu = \#$ extra photons observed in a measurement interval

α_{quantum}:

$$P_3 = \varepsilon_3 (1+\nu)$$

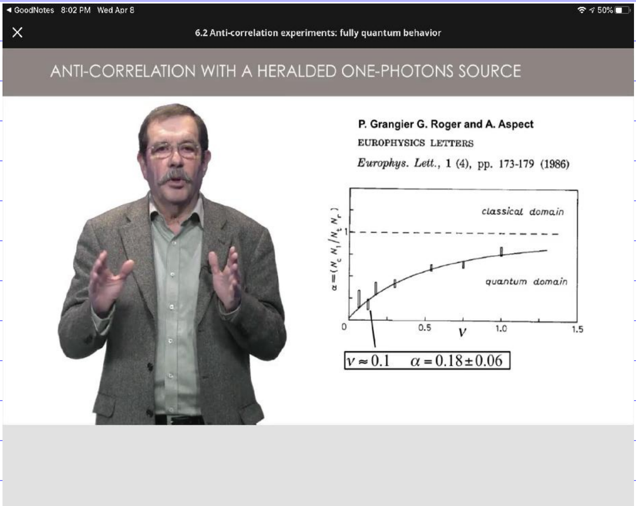
$$P_4 = \varepsilon_4 (1+\nu)$$

one photon events

$$P_c = \varepsilon_3 \varepsilon_4 (1+\nu)^2 - \varepsilon_3 \varepsilon_4$$

$$= \varepsilon_3 \varepsilon_4 (2\nu + \nu^2)$$

$$\alpha = \frac{2\nu + \nu^2}{(1+\nu)^2} = 1 - \frac{1}{(1+\nu)^2}$$



In summary, a one photon state, given by $|1\rangle = \sum_\ell c_\ell |1_\ell\rangle$ is an eigenstate of $\hat{N}$, but not necessarily $\hat{H}$ (it has definite particle number, but not necessarily definite energy. When we measure it, it can be observed only once. It has an extent in time given by some small multiple of $\frac{1}{\Gamma}$ with high probability. Sources of single photons are not just very dim light, as we see next.

We end the chapter with a discussion of semiclassical states of light, described by our old friend coherent states, which satisfy

$$\hat{a}_\ell |\alpha_\ell\rangle = \alpha_\ell |\alpha_\ell\rangle \qquad |\alpha_\ell\rangle = D(\alpha_\ell)|0\rangle = e^{\alpha_\ell^* \hat{a}_\ell + \alpha_\ell \hat{a}^\dagger}|0\rangle$$

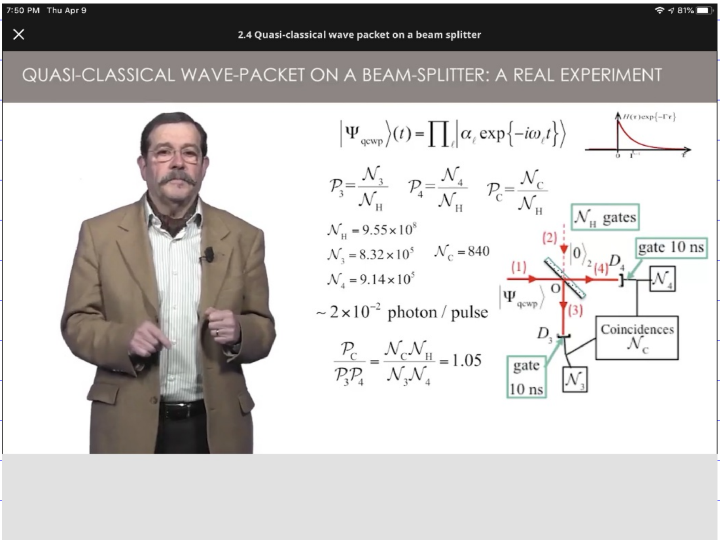
For a semiclassical state, we have

$$W^{(1)}(\vec{r},t) = s\,\|\hat{E}^+(\vec{r},t)\,|\alpha_\ell\rangle\|^2 = s\,(\mathcal{E}_\ell^{(1)})^2\,|\alpha_\ell|^2$$

$$W^{(2)}(\vec{r},t,\vec{r}',t') = s^2\,\|\hat{E}^+(\vec{r}',t')\,\hat{E}^+(\vec{r},t)\,|\alpha_\ell\rangle\|^2 = s^2\,(\mathcal{E}_\ell^{(1)})^4\,|\alpha_\ell|^4 = W^{(1)}(\vec{r},t)\,W^{(1)}(\vec{r}',t')$$

This is the classical result for a classical field as well

Note that it shows that even for very dim light, with much less than one photon in each measurement interval, we will sometimes observe two photons in one interval. Furthermore the "g" for semiclassical systems is $=1$. The experiment has been done and verified. This clarifies an old result where it was believed that classical light becomes a single photon source when very dim. But it never does.



Incandescent light, LED's, lasers, are all sources of semiclassical light. Essentially all light sources we commonly use are semiclassical.

Next time, we discuss squeezed light, measuring quadratures and how they improve LIGO.