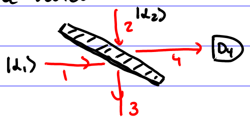
Lecture 35 How LIGO works

In this lecture, we will discuss how to measure the quadratures $\hat{Q}_\ell$ and $\hat{P}_\ell$, how to reduce the uncertainty in one value at the expense of the other, and how LIGO employs these ideas for higher precision.

To start, we note that one cannot directly measure an oscillating electric field of visible light because it oscillates too fast. The fastest oscilloscopes run at about 100 GHz, while light's oscillating frequency at $10^{14}-10^{15}$ Hz is 3-4 orders of magnitude faster. Nevertheless, there are schemes that do allow us to measure the fields of light by employing clever techniques. The first we will discuss is heterodyne detection. This involves measuring signals from two light beams, whose frequency differs by a small amount, and observing the beats of those signals, which oscillate much more slowly. Let's see how this works.

Heterodyne detection



In this experiment, we send in the weak light $|\alpha_1\rangle$ that we want to measure and the strong light $|\alpha_2\rangle$, used to "boost" the signal. We measure the photo current in channel 4.

$$|\psi_{in}\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle$$

$$i_4 = q_e \underset{\substack{\uparrow \\ \text{area}}}{S} w^{(1)}(r_4,t) = q_e S s \, \| E_4^{(+)}(r_4,t)|\psi_{in}\rangle\|^2$$

$$E_4^{(+)} = t E_1^{(+)} - r E_2^{(+)}$$

Assume the frequencies ω_1 and ω_2 are close, so that we can approximate

$$\mathcal{E}^{(1)}_{\omega_1} \sim \mathcal{E}^{(1)}_{\omega_2} \sim \mathcal{E}^{(1)}_{\omega} \qquad \omega = \frac{\omega_1+\omega_2}{2}$$

$$i_4 = q_e S s \left(\mathcal{E}^{(1)}_\omega\right)^2 \|(t \alpha_1 e^{-i\omega_1 t} - r \alpha_2 e^{-i\omega_2 t})|\psi_{in}\rangle\|^2$$

$$= q_e S s \left(\mathcal{E}^{(1)}_\omega\right)^2 \left\{ |t|^2|\alpha_1|^2 + |r|^2|\alpha_2|^2 - 2\,\mathrm{Re}\left(r t^* e^{i(\omega_1-\omega_2)t} \alpha_1^* \alpha_2\right) \right\}$$

let $\alpha_1 = |\alpha_1| e^{i\phi_1}$ $\quad \alpha_2 = |\alpha_2| e^{i\phi_2}$ assume r and t are real, then

$$i_4 = q_e S s \left(\mathcal{E}^{(1)}_\omega\right)^2 \left\{ |t|^2|\alpha_1|^2 + |r|^2|\alpha_2|^2 - 2rt|\alpha_1||\alpha_2| \cos\left[(\omega_1-\omega_2)t - \phi_1 + \phi_2\right] \right\}$$

recall $s\left(\mathcal{E}^{(1)}_\omega\right)^2 = \frac{\hbar}{ST}$ for light moving in a cylindrical quantization volume

recall as well that $|\alpha|^2$ is proportional to the number of photons in the quantization volume since $\langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$.

This number of photons increases as the length $L = cT$ increases by considering longer time intervals for the measurement. But Φ_{phot} = photon flux = $\frac{|\alpha|^2}{T}$ is independent of T (can think of as the "density" of photons)

The beam intensity, or energy density, is $\Phi = \Phi_{phot} \cdot \hbar\omega_\ell$

so the beam intensity satisfies $\frac{\Phi}{\hbar\omega_\ell} = \frac{|\alpha_\ell|^2}{T}$

or $|\alpha_\ell| = \sqrt{\frac{\Phi T}{\hbar\omega_\ell}}$

So the heterodyne signal becomes

$$i_4(t) = \eta q_e \left\{ t^2 \Phi_1^{phot} + r^2 \Phi_2^{phot} - 2rt\sqrt{\Phi_1^{phot}\Phi_2^{phot}} \cos\left[(\omega_1-\omega_2)t - \phi_1 + \phi_2\right]\right\}$$

If we tune r and t such that $t^2\Phi_1^{phot} = r^2\Phi_2^{phot}$

then $i_4(t) = 2t^2\Phi_1^{phot}\eta q_e\left(1 - \cos\left((\omega_1-\omega_2)t - \phi_1 + \phi_2\right)\right)$

which says the visibility is equal to 1.

The idea is that instead of measuring the small amplitude Φ_1^{phot}, we measure a much larger amplitude $\sqrt{\Phi_1^{phot}\Phi_2^{phot}}$.

But one must examine signal to noise to see if there is a true gain.

$i_{direct} = \eta q_e \Phi_1^{phot}$ $\qquad i_{hetero} = \eta q_e rt\sqrt{\Phi_1^{phot}\Phi_2^{phot}}$

(also called shot noise)

The noise is white noise. I don't have time to derive this here, but it is given by $\Delta i_{direct} = \sqrt{2 q_e i_{direct}\Delta f}$ $\qquad \Delta f = \frac{1}{T}$ = bandwidth

So $\left(\frac{\text{Signal}}{\text{Noise}}\right)_{direct} = \frac{i_{direct}}{\sqrt{2q_e i_{direct}\Delta f}} = \sqrt{\frac{i_{direct}}{2q_e\Delta f}} = \sqrt{\frac{\eta\Phi_1^{phot}}{2\Delta f}}$ } same

$\left(\frac{\text{signal}}{\text{Noise}}\right)_{hetero} = \frac{\eta q_e rt\sqrt{\Phi_1^{phot}\Phi_2^{phot}}}{r\sqrt{2q_e^2\eta\Phi_2^{phot}\Delta f}} = t\sqrt{\frac{\eta\Phi_1^{phot}}{2\Delta f}}$

q_e^2 because $i \propto q_e$ $\qquad$ Φ_2^{phot} dominates the noise

So there seems to be no gain. But this analysis ignored dark current noise. This is constant and can make it impossible to observe i_{direct}. So with dark current, one can measure signals via heterodyning that are much <u>smaller</u> than the dark current if $|\alpha_2|$ is larger.

Balanced homodyne detection.

This is very similar to heterodyning except now we take $\omega_1 = \omega_2$, $\vec{k}_1 = \vec{k}_2$, $\vec{\varepsilon}_1 = \vec{\varepsilon}_2$ so there is no oscillation.

We measure now the <u>difference</u> in photo currents $i_3 - i_4$

Note that the difference of two photo currents, measured over a long time interval, not the coincidences.

Recall that $sTw^{(1)}_{3,4}(r,t) = \eta\,|\langle\psi|\hat{N}_{3,4}|\psi\rangle|^2$

$$i_{3,4} = \eta\frac{q_e}{T}|\langle\psi|\hat{N}_{3,4}|\psi\rangle|^2$$

We need to express $\hat{N}_3 - \hat{N}_4$ in terms of the input creation/annihilation operators. In balanced homodyne detection, we take $r = t = \frac{1}{\sqrt{2}}$ so

$$\hat{N}_3 = (r\hat{a}_1^\dagger + t\hat{a}_2^\dagger)(r\hat{a}_1 + t\hat{a}_2)$$
$$= \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1)$$
$$\hat{N}_4 = (t\hat{a}_1^\dagger - r\hat{a}_2^\dagger)(t\hat{a}_1 - r\hat{a}_2)$$
$$= \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1)$$

So $\hat{N}_3 - \hat{N}_4 = \hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1$ we let $\alpha_2 = \alpha_{LO} = |\alpha_{LO}| e^{i\phi_{LO}}$

LO = local oscillator

$$i_3 - i_4 = \eta \frac{q_e}{T} \langle\psi_1|\hat{a}_1^\dagger|\psi_1\rangle |\alpha_{LO}| e^{i\phi_{LO}} + \langle\psi_1|\hat{a}_1|\psi_1\rangle |\alpha_{LO}| e^{-i\phi_{LO}}$$

$$= \eta \frac{q_e}{T} |\alpha_{LO}| \langle\psi_1|\hat{a}_1^\dagger e^{i\phi_{LO}} + \hat{a}_1 e^{-i\phi_{LO}}|\psi_1\rangle$$

$$= \eta \frac{q_e}{T} |\alpha_{LO}| \{ \langle\psi_1|\hat{a}_1^\dagger + \hat{a}_1|\psi_1\rangle \cos\phi_{LO} + i\langle\psi_1|\hat{a}_1^\dagger - \hat{a}_1|\psi_1\rangle \sin\phi_{LO} \}$$

recall $\hat{a}_1^\dagger + \hat{a}_1 \propto \hat{Q}_1$ and $i(\hat{a}_1^\dagger - \hat{a}_1) \propto \hat{P}_1$

So using balanced homodyne detection, we can measure the quadrature operators as functions of the local oscillator phase.

Note as well that the result is independent of the time interval T because the number of photons grows linearly in the time interval.

We need to now discuss fluctuations/noise.

$$(\hat{N}_3 - \hat{N}_4)^2 = (\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1)^2 = \hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_2\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_2\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_2^\dagger\hat{a}_1\hat{a}_1$$

put into normal ordered form (all †'s to the left) to find

$$= \hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_2\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_2^\dagger\hat{a}_1\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_1\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_2^\dagger\hat{a}_1\hat{a}_1$$

When taking the average with respect to $|\alpha_{LO}\rangle$ $\hat{a}_2 \to \alpha_{LO}$ $\hat{a}_2^\dagger \to \alpha_{LO}^*$

$$\langle\psi_{in}|(\hat{N}_3 - \hat{N}_4)^2|\psi_{in}\rangle = |\alpha_{LO}|^2 \{ \langle\psi_1|\hat{a}_1^\dagger\hat{a}_1^\dagger|\psi_1\rangle e^{+2i\phi_{LO}} + \langle\psi_1|\hat{a}_1\hat{a}_1|\psi_1\rangle e^{-2i\phi_{LO}} + \langle\psi_1|\hat{a}_1^\dagger\hat{a}_1|\psi_1\rangle + \langle\psi_1|\hat{a}_1\hat{a}_1^\dagger|\psi_1\rangle \} + \underbrace{\langle\psi_1|\hat{a}_1^\dagger\hat{a}_1|\psi_1\rangle}$$

neglect because small compared to $|\alpha_{LO}|^2$

$$\Rightarrow \left(\Delta(N_3 - N_4)\right)^2_{|\psi_{in}\rangle} = |\alpha_{LO}|^2 \{ \langle\psi_1|(\hat{a}_1^\dagger e^{i\phi_{LO}} + \hat{a}_1 e^{-i\phi_{LO}})^2|\psi_1\rangle - (\langle\psi_1|(\hat{a}_1^\dagger e^{i\phi_{LO}} + \hat{a}_1 e^{-i\phi_{LO}})|\psi_1\rangle)^2 \}$$

$$= |\alpha_{LO}|^2 \left(\Delta(\hat{a}_1^\dagger e^{i\phi_{LO}} + \hat{a}_1 e^{-i\phi_{LO}})\right)^2_{|\psi_1\rangle}$$

As mentioned before, this is related to the quadrature operators. We can define $\hat{Q}_1(\phi_{LO}) = \sqrt{\frac{\hbar}{2}}\left(\hat{a}_1^+ e^{i\phi_{LO}} + \hat{a}_1 e^{-i\phi_{LO}}\right)$

recalling $\hat{Q}_1 = \sqrt{\frac{\hbar}{2}}(\hat{a}_1 + \hat{a}_1^+)$ $\quad \hat{P}_1 = -i\sqrt{\frac{\hbar}{2}}(\hat{a}_1 - \hat{a}_1^+)$

says when $\phi_{LO} = 0$, we get Q, when $\phi_{LO} = \frac{\pi}{2}$ we get P

Note these are related to the electric field amplitude since

$$\hat{\vec{E}}(\vec{r},t) = \sum_\ell \vec{\varepsilon}_\ell \, \mathcal{E}_\ell^{(1)} \sqrt{\frac{2}{\hbar}} \left\{ -\hat{Q}_\ell \sin(\vec{k}_\ell\cdot\vec{r} - \omega_\ell t) - \hat{P}_\ell \cos(\vec{k}_\ell\cdot\vec{r} - \omega_\ell t) \right\}$$

Q is real part of amplitude $\quad$ P is imaginary part of amplitude

In general, we can consider $\hat{Q}(\theta)$ and $\hat{Q}(\theta + \frac{\pi}{2}) = \hat{P}(\theta)$. One can immediately verify that $[\hat{Q}(\theta), \hat{P}(\theta)] = i\hbar$

We will use these to measure the field. $\quad |\psi_1\rangle = |\alpha_1\rangle \quad \alpha_1 = |\alpha_1| e^{i\phi_1}$

$$\langle\psi_1|\hat{Q}(\theta)|\psi_1\rangle = \sqrt{\frac{\hbar}{2}} \langle\alpha_1|\hat{a}_1^+ e^{i\theta} + \hat{a}_1 e^{-i\theta}|\alpha_1\rangle$$
$$= \sqrt{2\hbar}\cos(\theta - \phi_1)|\alpha_1|$$
$$\langle\psi_1|\hat{P}(\theta)|\psi_1\rangle = -i\sqrt{\frac{\hbar}{2}} \langle\alpha_1| -\hat{a}_1^+ e^{i\theta} + \hat{a}_1 e^{-i\theta}|\alpha_1\rangle$$
$$= -\sqrt{2\hbar}\sin(\theta - \phi_1)|\alpha_1|$$

Variances:
$$\langle\psi_1|\hat{Q}(\theta)^2|\psi_1\rangle = \frac{\hbar}{2}\langle\alpha_1|\hat{a}_1^+\hat{a}_1^+ e^{2i\theta} + \hat{a}_1^+\hat{a}_1 + \hat{a}_1\hat{a}_1^+ + \hat{a}_1\hat{a}_1 e^{-2i\theta}|\alpha_1\rangle$$
$$= \hbar|\alpha_1|^2\left(\cos 2(\theta - \phi_1) + 1\right) + \frac{\hbar}{2}$$
$$= 2\hbar|\alpha_1|^2\cos^2(\theta - \phi_1) + \frac{\hbar}{2}$$
$$\langle\psi_1|\hat{P}(\theta)^2|\psi_1\rangle = -\frac{\hbar}{2}\langle\alpha_1|\hat{a}_1^+\hat{a}_1^+ e^{2i\theta} - \hat{a}_1^+\hat{a}_1 - \hat{a}_1\hat{a}_1^+ + \hat{a}_1\hat{a}_1 e^{-2i\theta}|\alpha_1\rangle$$
$$= -\hbar|\alpha_1|^2\left(\cos 2(\theta - \phi_1) - 1\right) + \frac{\hbar}{2}$$
$$= 2\hbar|\alpha_1|^2\sin^2(\theta - \phi_1) + \frac{\hbar}{2}$$

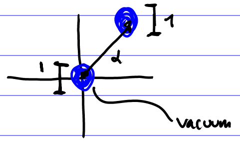
So $(\Delta Q(\theta))_{\psi_1} = \sqrt{\frac{\hbar}{2}} = (\Delta P(\theta))_{\psi_1}$

$(\Delta Q(\theta))_{\psi_1}(\Delta P(\theta))_{\psi_1} = \frac{\hbar}{2}$ = minimal uncertainty state

We can plot the uncertainty in a "phase space"

$$\text{Prob}(Q_1, P_1) = \frac{1}{2\pi\Delta Q_1^2} e^{-\frac{(Q_1 - \langle Q_1\rangle)^2 + (P - \langle P\rangle)^2}{2\Delta Q_1^2}}$$

Use reduced variables $\frac{Q}{\sqrt{2\hbar}}, \frac{P}{\sqrt{2\hbar}}$ $\quad \frac{2\Delta Q_1}{\sqrt{2\hbar}} = 1$



Displacement operator displaces from the vacuum to to α

Time evolution, given by $\alpha \to \alpha e^{-i\omega t}$, is a clockwise rotation in a circle. The uncertainty, being a circle, is always the same for Q and P.

Squeezed light trades off the uncertainty for Q to uncertainty in P or vice versa.

Recall Bogoliubov transformation from HW

$$\hat A_R = \cosh R\,\hat a + \sinh R\,\hat a^\dagger \qquad \hat A_R^\dagger = \cosh R\,\hat a^\dagger + \sinh R\,\hat a$$

$$[\hat A_R, \hat A_R^\dagger] = (\cosh^2 R - \sinh^2 R) = 1$$

consider the squeezed operators / states to be in one mode l only

A squeezed state satisfies $\hat A_R|\alpha, R\rangle = \alpha|\alpha, R\rangle$

Calculate averages by inverting $\hat a = \cosh R\,\hat A_R - \sinh R\,\hat A_R^\dagger$

$\hat a^\dagger = \cosh R\,\hat A_R^\dagger - \sinh R\,\hat A_R$

$$\langle \alpha, R|\hat E(\vec r, t)|\alpha, R\rangle = i\vec\varepsilon\,\mathcal{E}^{(1)}\left((\cosh R\,\alpha - \sinh R\,\alpha^*)e^{i(k\cdot r - \omega t)} + c.c.\right)$$

This is the same average as a QC state with $\alpha_{QC} \to \alpha\cosh R - \alpha^*\sinh R$
$= \mathrm{Re}\,\alpha\, e^{-R} + i\,\mathrm{Im}\,\alpha\, e^{R}$

If α is real, then $\alpha_{QC} = \alpha e^{-R}$

Calculating the variance is the same as before too, just use $\hat A_R$ and find $[\hat A_R, \hat A_R^\dagger]$ is what contributes so

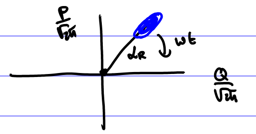
$$(\Delta E(0,t))^2_{|\alpha,R\rangle} = (\mathcal{E}^{(1)})^2\left[e^{2R}\cos^2(-\omega t) + e^{-2R}\sin^2(-\omega t)\right]$$

This takes a few lines to calculate, you should do it. Focus on only the $[\hat A_R, \hat A_R^\dagger]$ term

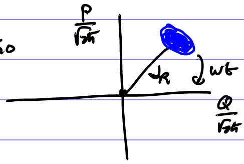
The variance changes as a function of time when $R \neq 0$.

The dispersion varies with time now

R>0



R<0

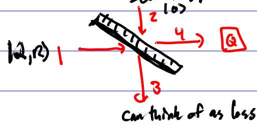


$R>0$ has the best accuracy when the field amplitude is large

$R<0$ has the best accuracy when the field amplitude is zero

the latter is best for measuring the phase as where the field crosses zero tells us the phases

When we measure on a beam splitter



loss is given by $t<1$, $r>0$

$$\hat Q_4 = \sqrt{\tfrac{\hbar}{2}}(\hat a_4 + \hat a_4^\dagger) = \sqrt{\tfrac{\hbar}{2}}\left(t\hat a_1 - r\hat a_2 + t\hat a_1^\dagger - r\hat a_2^\dagger\right)$$

$$= \sqrt{\tfrac{\hbar}{2}}\Big(t(\hat A_R\cosh R - \hat A_R^\dagger\sinh R) - r\hat a_2 + t(\hat A_R^\dagger\cosh R - \hat A_R\sinh R) - r\hat a_2^\dagger\Big)$$

$$= \sqrt{\tfrac{\hbar}{2}}\left(t\hat A_R e^{-R} + t\hat A_R^\dagger e^{-R} - r\hat a_2 - r\hat a_2^\dagger\right)$$

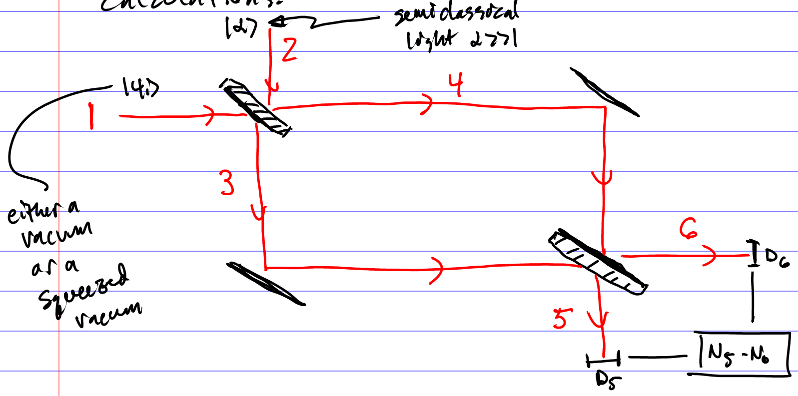
$$\langle\alpha R|\hat Q_4|\alpha R\rangle = \sqrt{\tfrac{\hbar}{2}}\, t e^{-R}(2\alpha) \quad \text{for } \alpha \text{ real}$$

$$\langle\alpha R|\hat Q_4^2|\alpha R\rangle - \left(\langle\alpha R|\hat Q_4|\alpha R\rangle\right)^2 = \frac{\hbar}{2}\left(t^2 e^{-2R} + \underbrace{r^2}_{\text{vac flucs of 2}}\right)$$

The vacuum fluctuations from channel 2 can ruin benefits of squeezing when one has losses.

Carlton Caves showed in 1980 how squeezing helps one measure on a Mach Zehnder interferometer.

I will give a brief description of how this works but not go through the detailed calculations.



Note how this is essentially a balanced homodyne detection at the lower right.

$$\langle \psi_{in} | \hat{N}_5 - \hat{N}_6 | \psi_{in} \rangle = \alpha \sqrt{\frac{2}{\hbar}} \langle \psi_i | \hat{P}_i | \psi_i \rangle$$

$$\langle \psi_{in} | (\hat{N}_5 - \hat{N}_6)^2 | \psi_{in} \rangle = \alpha^2 \frac{2}{\hbar} \langle \psi_i | \hat{P}_i^2 | \psi_i \rangle$$

So the fluctuations are determined by $(\Delta P_i)^2_{|\psi_i\rangle}$

If we use an ordinary vacuum $(\Delta P_i)^2_{|\psi_i\rangle} = \frac{\hbar}{2}$

but if we use a "p-squeezed vacuum" $(\Delta P_i)^2_{|\psi_i\rangle} = \frac{\hbar}{2} e^{2R}$

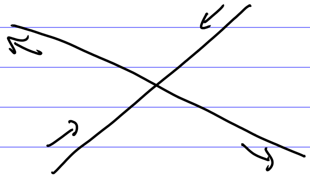
So with $R<0$, we get an improvement.

Note, the squeezed vacuum has photons in it. It is fragile and the gains are reduced with losses so one needs super high quality mirrors and optics.

The improvement of accuracy by $1+\delta$, will increase the volume of observed universe by $(1+\delta)^3$ so even small improvements will create huge increases in the observable universe with gravity waves.

LIGO is an interferometer

Gravity waves are quadrupole waves



like a "d-wave" excitation

Each arm is 4km long and detects $\frac{\delta L}{L} \sim 10^{-21}$

or a measurement of 10^{-18} m ($\frac{1}{1000}$ the radius of a nucleus)

Resonant cavities are used. These increase the effective length by a factor of 300. Interferometry can measure $\sim$ 10 nm ($\lambda \sim 500$ nm).

so the length difference is $\frac{\delta L}{L} = \frac{10 \times 10^{-9}\,m}{300 \cdot 4000\,m} \sim 10^{-14}$

I believe the other 7 orders of magnitude come from the $|\alpha|^2$ from the lasers, but I have not been able to confirm this.

The squeezed vacuum reduces noise by 28%

$\Rightarrow$ $(1.28)^3 \sim$ twice as much universe can be observed