Coherent states play two important roles. On the one hand, they can be used to illustrate how close a quantum system can be made to look like the back-and-forth motion of a mass on a spring. On the other hand, they play a critical role in the quantization of light, with coherent states being the quantum representation of the electromagnetic fields. We won’t be able to go into detail about that here, but we will develop these states further. They are our connection to the classical world.

There are many ways to motivate the coherent states, but here, we do so as the eigenstates of the lowering operator
\[ \hat{a} \mid \psi \rangle = \sqrt{n} \mid \psi \rangle. \]

But wait, you say, \( \hat{a} \) is not Hermitian, how can it have eigenstates? Well, turns out it can, but some of our familiar properties do not hold. First, \( \hat{a} \) can be complex and need not be real and scalar, the eigenstates are not orthogonal. They also are not complete, which we will describe below.

But first, let’s refresh our memories about \( \hat{a} \).

\[
\begin{align*}
\{ \hat{a}, \hat{a}^\dagger \} &= 2n \quad \{ \hat{a}^\dagger \hat{a} + \frac{1}{2} \hat{1}, \hat{a} \hat{a}^\dagger \} = -2n \quad \hat{a}^\dagger \hat{a} \hat{a} \hat{a}^\dagger &= -\frac{1}{2} n^2 \\
\{ \hat{a}, \hat{a}^\dagger \} &= 2n \quad \{ \hat{a}^\dagger + \hat{a} + \frac{1}{2} \hat{1}, \hat{a} \hat{a}^\dagger \} = 2n \quad \hat{a}^\dagger \hat{a} &= 2n^2 
\end{align*}
\]

Now, let us conclude that \( \hat{a} \) lowers the energy of the energy eigenstates by \( 2n \), while \( \hat{a}^\dagger \) raises it by \( \frac{1}{2} n^2 \).

We also know
\[
\begin{align*}
\{ \hat{a}, \hat{a}^\dagger \} &= -\frac{1}{2} \quad \{ \hat{a}^\dagger \hat{a} + \frac{1}{2} \hat{1}, \hat{a} \hat{a}^\dagger \} = -n \quad \hat{a}^\dagger \hat{a} \hat{a} \hat{a}^\dagger &= \frac{n}{2} \\
\{ \hat{a}, \hat{a}^\dagger \} &= -\frac{1}{2} \quad \{ \hat{a}^\dagger + \hat{a} + \frac{1}{2} \hat{1}, \hat{a} \hat{a}^\dagger \} = -\frac{n}{2} \quad \hat{a}^\dagger \hat{a} &= -\frac{n}{2}
\end{align*}
\]

(If you don’t, then use induction to show it)

So
\[
\begin{align*}
\hat{a} \mid n \rangle &= \hat{a} \left( \frac{\hat{a}^\dagger \hat{a}}{\sqrt{n}} \right) \mid 0 \rangle \\
&= \left( \frac{\hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a}}{\sqrt{n}} \right) \mid 0 \rangle \\
&= \frac{n}{\sqrt{n}} \mid n-1 \rangle
\end{align*}
\]

Similarly \( \hat{a}^\dagger \mid n \rangle = \sqrt{n} \mid n+1 \rangle \) (but this proof is much easier since it is a matrix that is real).

Let’s find \( \mid d \rangle \). Let \( \mid d \rangle = \sum_{n=0}^{\infty} C_n \mid n \rangle \). (Possible since it is complete)

\[
\begin{align*}
\hat{a} \mid d \rangle &= \sum_{n=0}^{\infty} C_n \hat{a} \mid n \rangle = \sum_{n=0}^{\infty} C_n \hat{a} \sqrt{n} \mid n \rangle \\
&= \hat{d} \mid d \rangle = \sum_{n=0}^{\infty} C_n \sqrt{n} \mid n \rangle \\
&= \hat{d} \mid d \rangle \implies C_n \sqrt{n} = \hat{d} C_n
\end{align*}
\]
$c_1 = \frac{1}{\sqrt{2}} c_0$, $c_2 = \frac{1}{\sqrt{2}} c_0$, $c_3 = \frac{j}{\sqrt{2}} c_0$, $c_4 = \frac{-j}{\sqrt{2}} c_0$.

And $C_n = \frac{d^n}{\sqrt{n!}} c_0$.

$c_0$ is determined by normalization. Let's first compute the overlap $\langle \beta \mid \alpha \rangle$:

$$\langle \beta \mid \alpha \rangle = \sum_{n=0}^{\infty} \frac{(\beta^n)^n}{\sqrt{n!}} c_0^n \langle n \mid \sum \frac{\alpha^n}{\sqrt{n!}} c_0^n \rangle$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\beta^m)^n}{\sqrt{n! \sqrt{m!}}} c_0^n \langle n \mid c_0^m \rangle$$

Set $\alpha = \beta$ so:

$$\langle \alpha \mid \alpha \rangle = e^{\frac{1}{2} |\alpha|^2} |c_0 \rangle \langle c_0|^2 = 1$$

$$|c_0 \rangle = e^{-\frac{1}{2} |\alpha|^2} \text{ (pick } c_0 \text{ real and positive)}$$

So

$$\langle \alpha \mid \alpha \rangle = \sum_{n=0}^{\infty} \frac{d^n}{\sqrt{n!}} e^{-\frac{1}{2} |\alpha|^2} |n \rangle \langle n| = e^{\frac{1}{2} |\alpha|^2} e^{-\frac{1}{2} |\alpha|^2} |0 \rangle$$

Note that the overlap is non-zero, but as $\alpha \beta$ becomes large in magnitude, $\langle \alpha \mid \alpha \rangle$ gets small very quickly. Hence coherent states are not orthogonal!

You should think about why $\hat{\alpha}$ has no eigenstate. Ask if you do not see why it cannot work.

Let us play with the representation of a coherent state

$$|\alpha \rangle = \sum_{n=0}^{\infty} \frac{d^n}{\sqrt{n!}} e^{-\frac{1}{2} |\alpha|^2} |n \rangle = e^{-\frac{1}{2} |\alpha|^2} \sum_{n=0}^{\infty} \frac{d^n}{\sqrt{n!}} |0 \rangle$$

but recall $e^{-\hat{\alpha}^*} |0 \rangle = |0 \rangle$ so (recall $|\alpha|^2 = 2$)

$$|\alpha \rangle = e^{-\frac{1}{2} |\alpha|^2} e^{\hat{\alpha}} e^{-\hat{\alpha}} |0 \rangle$$

Now use BEC with $\hat{\alpha} = e^{\hat{\alpha}_+} \hat{\alpha} = -e^{\hat{\alpha}_-}$ $[\hat{\alpha}_+, \hat{\alpha}_-] = 1 |\alpha \rangle \langle \alpha |$

$$|\alpha \rangle = e^{-\frac{1}{2} |\alpha|^2} e^{\hat{\alpha}_+} e^{\hat{\alpha}_-} |0 \rangle$$

We define the displacement operator $\hat{D}(\alpha)$ to be $\hat{D}(\alpha) = e^{\hat{\alpha}_+ \hat{\alpha}_-}$

$$|\alpha \rangle = \hat{D}(\alpha) |0 \rangle$$
But recall that
\[
\hat{\alpha} = \frac{1}{\sqrt{2m_0 \hbar}} (\hat{x} + \frac{i}{\hbar \gamma} \hat{p}) \quad \hat{a}^* = \frac{1}{\sqrt{2m_0 \hbar}} (\hat{x} - \frac{i}{\hbar \gamma} \hat{p})
\]

so that \( \hat{a}^* \hat{a} - \hat{a} \hat{a}^* = \frac{1}{\hbar \gamma} \), \( \text{Im} \hat{x} - \frac{i}{\hbar \gamma} \hat{p} = \hat{a}^* \hat{a} - i \frac{\hat{p}^2}{\hbar \gamma} \text{Re} \hat{p} \)

So, if \( \hat{a} \) is real, \( \hat{a}^* \hat{a} = i \frac{\hat{p} \hat{x}}{\hbar \gamma} \quad \hat{x} \hat{a} = \frac{\hat{p}}{\hbar \gamma} \text{Re} \hat{p} \)

If \( \hat{a} \) is imaginary, \( \hat{a}^* \hat{a} = i \frac{\hat{p} \hat{x}}{\hbar \gamma} \quad \hat{p} \hat{a} = \frac{\hat{x}}{\hbar \gamma} \text{Im} \hat{p} \)

\( \Rightarrow \) the displacement operator gives both a translation in space and a translation in momentum to the ground state energy eigenfunction.

Calculating with coherent states is easy because
\( \hat{a} |\psi\rangle = \lambda |\psi\rangle \) and \( \langle \psi | \hat{a}^* = \langle \psi | \lambda \)

So
\[
\langle \psi | \hat{x} |\psi\rangle = \frac{\lambda}{2m_0 \hbar} \quad \langle \psi | \hat{a}^* \hat{a} + a^* a |\psi\rangle = \frac{\lambda^2}{2m_0 \hbar} \text{Re} \lambda
\]

\[
\langle \psi | \hat{x}^2 |\psi\rangle = \frac{\lambda^2}{2m_0 \hbar} \langle \psi | \hat{a}^* \hat{a} + \hat{a} \hat{a}^* + a^* a + a a^* |\psi\rangle
\]
\[
= \frac{\hbar}{2 \sqrt{2 \pi \hbar}} \langle \psi | \hat{a}^* \hat{a} + \hat{a} \hat{a}^* + [\hat{a}, \hat{a}^*] + \hat{a}^* \hat{a} + a^* a + a a^* |\psi\rangle
\]
\[
= \frac{\hbar}{2 \sqrt{2 \pi \hbar}} \left( \lambda^2 + 2 \lambda^2 + 1 + \lambda^2 \right)
\]

So
\[
(\Delta x)^2_x = \frac{\lambda^2}{2m_0 \hbar} \quad \text{independent of } \lambda \quad (\text{coherent state shape is independent of the coherence})
\]

Displacing the ground state does not change its uncertainty:
\[
\langle \psi | \hat{x} |\psi\rangle = -i \frac{\hbar}{2 \sqrt{m_0 \hbar}} \langle \psi | \hat{a}^* - \hat{a} |\psi\rangle = -i \frac{\hbar}{2 \sqrt{m_0 \hbar}} (\lambda - \lambda^*)
\]

\[
\langle \psi | \hat{x}^2 |\psi\rangle = \frac{\hbar}{2 \sqrt{2 \pi \hbar}} \langle \psi | \hat{a}^* \hat{a} + \hat{a} \hat{a}^* - [\hat{a}, \hat{a}^*] + a^* a + a a^* |\psi\rangle
\]
\[
= -\frac{\hbar}{2 \sqrt{2 \pi \hbar}} \left( \lambda^2 - 2 \lambda^2 + 1 + \lambda^2 \right)
\]

So
\[
(\Delta p)^2_x = \frac{\hbar^2}{2 \sqrt{2 \pi \hbar}} \quad \Rightarrow \quad (\Delta p)_x (\Delta x)_x = \frac{\hbar}{2}
\]

The uncertainty is unchanged in a coherent state!
What about their time dependence?

We have not discussed time dependence in general yet, but one should see immediately that

$$i \hbar \frac{\partial}{\partial t} \psi(t) = H(\psi(t))$$

is solved by

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = e^{-i \frac{\hat{H}}{\hbar} t} |\psi(0)\rangle$$

Just check by taking the derivative. Because $\hat{H}$ is independent of time, there are no operator ordering issues.

So, we have

$$|\psi(t)\rangle = e^{-i \frac{\hat{H}}{\hbar} t} |\psi(0)\rangle$$

But $\hat{H} = \hbar \omega (\hat{a}^+ \hat{a} + \frac{1}{2})$ so

$$|\psi(t)\rangle = e^{-i \hbar \omega (\hat{a}^+ \hat{a} + \frac{1}{2}) t} |\psi(0)\rangle$$

$$= e^{-i \hbar \omega (\hat{a}^2 + \frac{1}{2}) t} e^{\frac{i}{\hbar} \omega \hat{a}^+ \hat{a} |\psi(0)\rangle}$$

How do we proceed? We need to move the time operators through each other. This is exponential reordering, or the braiding identity — we get

$$|\psi(t)\rangle = \exp \left[ e^{-i \hbar \omega (\hat{a}^2 + \frac{1}{2}) t} (e^{i \hbar \omega \hat{a}^+ \hat{a}}) e^{-i \hbar \omega (\hat{a}^2 + \frac{1}{2}) t} \right]$$

$$= e^{-i \frac{\hbar \omega}{2} t} |\psi(0)\rangle$$

since $\hat{a} \hat{a} \hat{a} = 0$

So we need to compute

$$\hat{U}(t) \hat{a} \hat{U}^+(t) \quad \text{and} \quad \hat{U}(t) \hat{a}^+ \hat{U}^+(t)$$

The easiest way to do this is by differentiating (note this does not always work, but does so here because of the simplicity of $[\hat{a}, \hat{a}^+]$ and $[\hat{a}^+, \hat{a}^+]$)

$$\frac{d}{dt} \left( \hat{U}(t) \hat{a} \hat{U}^+(t) \right) = \left( \frac{d}{dt} \hat{U}(t) \right) \hat{a} \hat{U}^+(t) + \hat{U}(t) \left( \hat{a} \frac{d}{dt} \hat{U}^+(t) \right)$$

$$= -i \hbar \omega \hat{U}(t) (\hat{a} \hat{a} \hat{a} - \hat{a} \hat{a} \hat{a}^+ \hat{a}^+) \hat{U}^+(t)$$

$$= -i \hbar \omega \hat{U}(t) (\hat{a} \hat{a} \hat{a}^+ \hat{a}^+) \hat{U}^+(t)$$

$$= i \hbar \omega \hat{U}(t) \hat{a} \hat{U}^+(t)$$

$$\Rightarrow \quad \frac{d}{dt} \hat{U}(t) \hat{a} \hat{U}^+(t) = e^{i \hbar \omega \hat{a} \hat{a}^+ \hat{a}^+}$$

Similarly

$$\frac{d}{dt} \hat{U}(t) \hat{a}^+ \hat{U}^+(t) = e^{-i \hbar \omega \hat{a} \hat{a}^+ \hat{a}^+}$$

because of the same change in the commutation...
Hence
\[
\langle \psi(t) \rangle = \exp \left[ -\frac{1}{2} \omega t a^* a \right] e^{-i \omega t} \langle 0 \rangle \\
= e^{-\frac{1}{2} \omega t} \langle 0 \rangle
\]

3) The coherent state evolves in time simply via its parameter $\alpha$ being multiplied by $e^{-i \omega t}$. This is very simple.

Let's try to understand by computing expectation value at position as a function of time
\[
\langle x(0) \rangle = \int x \left| \frac{\exp \left[ -\frac{1}{2} \omega t a^* a \right] e^{-i \omega t}}{\sqrt{2\pi \hbar \omega}} \right|^2 dx
\]
\[
= \int \frac{1}{\sqrt{2\pi \hbar \omega}} \left( e^{-\frac{1}{2} \omega t} e^{-i \omega t} \right) dx
\]
\[
= \frac{1}{\sqrt{2\pi \hbar \omega}} \left( x_0 \cos \omega t + P_0 \sin \omega t \right)
\]

This is the classical equation of motion for a spring!

Similarly
\[
\langle \hat{p}(0) \rangle = -i \sqrt{\frac{\hbar \omega}{2}} \int \left| \frac{\exp \left[ -\frac{1}{2} \omega t a^* a \right] e^{-i \omega t}}{\sqrt{2\pi \hbar \omega}} \right|^2 \hat{a} dx
\]
\[
= -i \sqrt{\frac{\hbar \omega}{2}} \left( e^{-\frac{1}{2} \omega t} e^{-i \omega t} \right) \hat{a}
\]
\[
= -i \sqrt{\frac{\hbar \omega}{2}} \left( -i \sqrt{\frac{\hbar \omega}{2}} \right) \left( x_0 \sin \omega t + P_0 \cos \omega t \right)
\]

\[
\langle \hat{p}(0) \rangle = -\hbar \omega x_0 \sin \omega t + P_0 \cos \omega t
\]

also the classical equation of motion!

You will show on a homework exercise that the uncertainty is independent of time as well.

So we can think of the coherent state as being a "blob" whose uncertainty remains the same for all time and it sloshes back and forth as a classical mass on a spring.

This is as close to a classical image we got with quantum systems.
What are the probabilities to observe the system to have different energies when it is prepared in a coherent state.

\[
|\langle n | \psi \rangle|^2 = \prod_{n=0}^{\infty} \frac{a^n e^{-\frac{a^2}{2}} (\frac{a}{\hbar})^n}{n!}^2
\]

\[P(n) = \frac{1}{n!} e^{-1/2} \frac{(1/2)^n}{n!} \]

This implies that for any nonzero \( n \), there is a nonzero probability to see any energy excitations.

By differentiating this we find the maximum occurs when \[
d\frac{1}{n!} e^{-1/2} \frac{1}{n!} \left( \frac{1/2}{n!} \right) = 0 \quad \Rightarrow \quad \frac{1}{n!} - 1 = 0 \quad \Rightarrow \quad n = 1
\]

Hence, this maximal probability increases with \( n \).

When we later discuss photons, we will see we can think of \( n \) as being related to the number of photons and classical sources of light (light bulbs and even lasers) produce light in coherent states.

So no matter how dim the light is, it is never a single photon source — there is always the possibility of photon bunching.

An experiment was done of a two-slit experiment with light dimmed so low that it took 8 months for enough light to expose the photographic plate. It did show a two-slit interference pattern.

But this experiment used a coherent state source not a single photon source so it could put out multiple photons being in the apparatus at the same time. Indeed, it is certain that this did occur due to photon bunching. But quantum theory was not well enough developed at that time for anyone to know this was the case.