Suppose we have two angular momenta which are independent and commute with each other

$\hat{J}_1$ and $\hat{J}_2$

$$[\hat{J}_{1i}, \hat{J}_{1j}] = i\hbar \varepsilon_{ijk} \hat{J}_{1k} \qquad [\hat{J}_{2i}, \hat{J}_{2j}] = i\hbar\varepsilon_{ijk} \hat{J}_{2k}$$

$$[\hat{J}_{1i}, \hat{J}_{2j}] = 0$$

Form the total angular momentum operator

$$\hat{J} = \hat{J}_1 + \hat{J}_2$$

Then
$$[\hat{J}_i, \hat{J}_j] = [\hat{J}_{1i} + \hat{J}_{2i}, \hat{J}_{1j} + \hat{J}_{2j}]$$
$$= i\hbar \varepsilon_{ijk} (\hat{J}_{1k} + \hat{J}_{2k})$$
$$= i\hbar \varepsilon_{ijk} \hat{J}_k$$

claim: $\hat{J}_1^2$, $\hat{J}_2^2$ $\hat{J}_{1z}$ and $\hat{J}_{2z}$ all commute

proof - obvious since 1 & 2 commute and
$$[\hat{J}_1^2, \hat{J}_{1z}] = [\hat{J}_2^2, \hat{J}_{2z}] = 0$$

so we can form eigenstates as $|j_1, m_1, j_2, m_2\rangle$ such that

$$\hat{J}_1^2 |j_1, m_1, j_2, m_2\rangle = \hbar^2 j_1(j_1+1) |j_1 m_1 j_2 m_2\rangle$$
$$\hat{J}_2^2 |j_1 m_1 j_2 m_2\rangle = \hbar^2 j_2 (j_2+1) |j_1 m_1 j_2 m_2\rangle$$
$$\hat{J}_{1z} |j_1 m_1 j_2 m_2\rangle = \hbar m_1 |j_1 m_1 j_2 m_2\rangle$$
$$\hat{J}_{2z} |j_1 m_1 j_2 m_2\rangle = \hbar m_2 |j_1 m_1 j_2 m_2\rangle$$

we also have

$\hat{J}^2, \hat{J}_z, \hat{J}_1^2, \hat{J}_2^2$ all commute

proof: $[\hat{J}_z, \hat{J}_1^2] = 0 \quad [\hat{J}_z, \hat{J}_2^2] = 0 \quad [\hat{J}^2, \hat{J}_z] = 0$

need to check

$$[\hat{J}^2, \hat{J}_1^2] = 0$$

But
$$\hat{J}^2 = \hat{J}_1^2 + 2\hat{J}_1 \cdot \hat{J}_2 + \hat{J}_2^2$$
$$= \hat{J}_1^2 + 2 J_{1j} J_{2j} + \hat{J}_2^2$$

but $[\hat{J}_1^2, \hat{J}_{1j}] = 0 \Rightarrow [\hat{J}^2, \hat{J}_1^2] = 0$

similarly $[\hat{J}^2, \hat{J}_2^2] = 0$

so we also can label states by $j\, m_j\, j_1\, j_2$

$$\hat{J}^2 |j m_j j_1 j_2\rangle = \hbar^2 j(j+1) |j m_j j_1 j_2\rangle$$
$$\hat{J}_z |j m_j j_1 j_2\rangle = \hbar m_j |j m_j j_1 j_2\rangle$$
$$\hat{J}_1^2 |j m_j j_1 j_2\rangle = \hbar^2 j_1(j_1+1) |j m_j j_1 j_2\rangle$$
$$\hat{J}_2^2 |j m_j j_1 j_2\rangle = \hbar^2 j_2(j_2+1) |j m_j j_1 j_2\rangle$$

Both representations are completely equivalent and span the whole basis of states.

Questions: 1) given a j_1 and j_2 what values of j are allowed?

2.) how do I convert between the different bases?

The coefficients of the expansions are called Clebsch-Gordon coefficients.

Suppose we start with two states, one with j_1 and one with j_2 and those values are fixed.

Look at the representation $|j_1 m_1\ j_2 m_2\rangle$

We have

$$\hat{J}_z |j_1 m_1\ j_2 m_2\rangle = (\hat{J}_{1z} + \hat{J}_{2z}) |j_1 m_1\ j_2\ m_2\rangle$$

$$= \hbar\,(m_1 + m_2)\, |j_1 m_1\ j_2\ m_2\rangle$$

$\Rightarrow$ This representation is an eigenfunction of $\hat{J}_z$ with eigenvalue $m_j = m_1 + m_2$.

In general, this state is not an eigenstate of $\hat{J}^2$.

examine the maximal spin state

$$|j_1\ m_1 = j_1\ j_2\ m_2 = j_2\rangle$$

Then $m = j_1 + j_2$. This is the maximal m_j we can have, so the maximal j we can have is $m_1 + m_2$

In other words, this state is also

$$|j = j_1 + j_2\ \ m_j = j_1 + j_2\ \ j_1\ j_2\rangle$$

Up to a phase.

Similarly, the state

$$|j_1\ m_1 = -j_1\ \ j_2\ m_2 = -j_2\rangle$$

is $\quad |j = j_1 + j_2\ \ m_j = -j_1 - j_2\ \ j_1\ j_2\rangle$ up to a phase.

How do we find the state with

$$j = j_1 + j_2 - 1 \quad ?$$

look at the states with $m_j = j_1 + j_2 - 1$

$$|j_1\, m_1 = j_1 - 1\; j_2\, m_2 = j_2\rangle \text{ and } |j_1\, m_1 = j_1\; j_2\, m_2 = j_2 - 1\rangle$$

these states must have one linear combination

which has $j = j_1 + j_2$ and $m_j = j_1 + j_2 - 1$

and one which has

$j = j_1 + j_2 - 1$ and $m_j = j_1 + j_2 - 1$

How to find them?

recall $\hat{J}_- |j\, m_j = j\; j_1\, j_2\rangle \propto |j\, m_j = j-1\; j_1\, j_2\rangle$

so we find this state by hitting with $\hat{J}_-$ and

the state with $j = j_1 + j_2 - 1$ is orthogonal to this

state.

Similarly, ~~so~~ if we look at $M_j = j_1 + j_2 - 2$, there

are three states
$m_1 = j_1 \quad m_2 = j_2 - 2$
$m_1 = j_1 - 1 \quad m_2 = j_2 - 1$
$m_1 = j_1 - 2 \quad m_2 = j_2$

and we will find the $j = j_1 + j_2$, $j = j_1 + j_2 - 1$ and

$j = j_1 + j_2 - 2$ states in this subspace

and so on.

Now suppose $j_1 > j_2$ biggest m is $j_1 + j_2 \Rightarrow$ biggest $j = j_1 - j_2$

but the smallest j is not necessarily 0 or $\frac{1}{2}$

it is actually $|j_1 - j_2| = j_{min}$

let's see why by example: $j_1 = 2$ $j_2 = 1$

m	m_1	m_2	
3	2	1	$j = 3$
2	2	0	$j = 3$ and 2
	1	1	
1	2	-1	$j = 3, 2$ and 1
	1	0	
no -1, 2	0	1	
0	1	-1	$j = 3, 2$ and 1 no 0 !
	0	0	
	-1	1	
-1	0	-1	$j = 3, 2$ and 1
	-1	0	
	-2	1	
-2	-1	-1	$j = 3,$ and 2
	-2	0	
-3	-2	-1	$j = 3$

We can also do counting to check

~~$\sum_{j=j_{min}}^{j_{max}} (2j+1)$ = total number of states in $|j m\rangle$ rep~~ $j_1 j_2\rangle$

$(2j_1+1)(2j_2+1)$ = total # of states in $|j_1 m_1\, j_2 m_2\rangle$ rep

but $\displaystyle\sum_{j=j_{min}}^{j_{max}} (2j+1) = 2\cdot\left(\frac{(j_{max}+1)\,j_{max}}{2} - \frac{j_{min}(j_{min}-1)}{2}\right) + j_{max} - j_{min} + 1$

$\displaystyle\sum_{j=0}^{n} j = \frac{(n+1)n}{2}$

$$= (j_1+j_2+1)(j_1+j_2) - j_{min}(j_{min}-1) + j_1 + j_2 - j_{min} + 1$$

$$= 4j_1j_2 + 2j_1 + 2j_2 + 1$$

subtract

$$= j_1^2 + 2j_1j_2 + j_2^2 + j_1 + j_2 - j_{min}(j_{min}-1) + j_1 + j_2 - j_{min} + 1$$

$$-4j_1j_2 - 2j_1 - 2j_2 - 1$$

$$= j_1^2 - 2j_1j_2 + j_2^2 - j_{min}(j_{min}) = 0$$

$$j_{min}^2 - (j_1 - j_2)^2 = 0$$

$$\Rightarrow j_{min} = |j_1 - j_2| \quad ✓$$

So we write

$$|j m\, j_1 j_2\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 m_1\, j_2 m_2\rangle \langle j_1 m_1\, j_2 m_2 | j m\, j_1 j_2\rangle$$

Clebsch-Gordon coefficients

Similarly

$$|j_1 m_1\, j_2 m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} |j m\, j_1 j_2\rangle \langle j m\, j_1 j_2 | j_1 m_1\, j_2 m_2\rangle$$

complex conjugate
also called a
Clebsch-Gordon coefficient

phases chosen so that coefficients are real,
so only ambiguity is in $\pm$ signs which are
fixed by a convention.