Addition of Angular momentum II

Let's examine the simplest case concretely

$j_1 = \frac{1}{2} \qquad j_2 = \frac{1}{2}$

We can form ~~an~~ a $J = 1$ state (called a triplet)
and a $J = 0$ state (called a singlet)

Obviously, we have $j_1 = j_2 = \frac{1}{2}$ always

$|j, m_J, j_1 j_2\rangle = |j\, m_j\rangle$

can be written as

$$|j=1\, m_j = +1\rangle = |m_1 = +\tfrac{1}{2}, m_2 = +\tfrac{1}{2}\rangle = |\uparrow\uparrow\rangle$$

since this is the only way to get $+1$ for m_j

and

$$|j=1, m_j = -1\rangle = |m_1 = -\tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle = |\downarrow\downarrow\rangle$$

but for $m_j = 0$ we have two possibilities

$$|j=1, m_j = 0\rangle = \alpha\, |m_1 = \tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle + \beta\, |m_1 = -\tfrac{1}{2}, m_2 = +\tfrac{1}{2}\rangle$$

$$= \alpha\, |\uparrow\downarrow\rangle + \beta\, |\downarrow\uparrow\rangle$$

How do we find α and β?

Answer: use the total spin lowering operator.

We know

$$J^- \,|j=1, m_j=1\rangle = \hbar\sqrt{(j+m_j)(j-m_j+1)}\;|j=1, m_j=0\rangle$$

$$= \hbar\sqrt{2\cdot 1}\;|j=1, m_j=0\rangle$$

$$= \sqrt{2}\,\hbar\,|j=1, m_j=0\rangle$$

But $J^- = J_1^- + J_2^- = S_1^- + S_2^-$

$$S_i^- \,|\uparrow\rangle = \hbar\,|\downarrow\rangle$$

so $$(S_1^- + S_2^-)\,|\uparrow\uparrow\rangle = \hbar\,|\downarrow\uparrow\rangle + \hbar\,|\uparrow\downarrow\rangle$$

$$= \sqrt{2}\,\hbar\,|j=1, m_j=0\rangle$$

so $\boxed{\alpha = \frac{1}{\sqrt{2}} \text{ and } \beta = \frac{1}{\sqrt{2}}}$ $\boxed{|j=1, m_j=0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle}$

How to find $|j=0, m_j=0\rangle$?

It must be orthogonal to $|j=1, m_j=0\rangle$ since

$$\langle j=1, m_j=0 | j=0, m_j=0\rangle = 0$$

so it must be

$$\boxed{|j=0, m=0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle}$$

Now lets work this out for the general case

j_1 and $j_2 = \frac{1}{2}$

$j = j_1 + \frac{1}{2}$ or $j_1 - \frac{1}{2}$

Obviously

$$|j = j_1 + \tfrac{1}{2}, m_j = j_1 + \tfrac{1}{2}\rangle = |m_1 = j_1, m_2 = \tfrac{1}{2}\rangle$$

To find

$$|j = j_1 + \tfrac{1}{2}, m_j = j_1 - \tfrac{1}{2}\rangle$$

Use the lowering operator, recalling that

$$J^- |j = j_1 + \tfrac{1}{2}, m_j = j_1 + \tfrac{1}{2}\rangle = \hbar\sqrt{(j_1 + \tfrac{1}{2} + j_1 + \tfrac{1}{2})(j_1 + \tfrac{1}{2} - j_1 - \tfrac{1}{2} + 1)}\, |j = j_1 + \tfrac{1}{2}, m_j = j_1 - \tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{2j_1 + 1}\, |j = j_1 + \tfrac{1}{2}, m_j = j_1 - \tfrac{1}{2}\rangle$$

But

$$J^- = J_1^- + S_2^- \quad \text{so}$$

$$J^- |m_1 = j_1, m_2 = \tfrac{1}{2}\rangle = \hbar\sqrt{(2j_1)\cdot(1)}\, |m_1 = j_1 - 1, m_2 = \tfrac{1}{2}\rangle$$

$$+ \hbar\, |m_1 = j_1, m_2 = -\tfrac{1}{2}\rangle$$

so

$$\boxed{|j = j_1 + \tfrac{1}{2}, m_j = j_1 - \tfrac{1}{2}\rangle = \frac{1}{\sqrt{2j_1+1}}\, |m_1 = j_1, m_2 = -\tfrac{1}{2}\rangle + \sqrt{\frac{2j_1}{2j_1+1}}\, |m_1 = j_1 - 1, m_2 = \tfrac{1}{2}\rangle}$$

The state $|j_1 = j_1 - \frac{1}{2}, m_j = j_1 - \frac{1}{2}\rangle$ is orthogonal to this so

$$\boxed{|j = j_1 - \tfrac{1}{2}, m_j = j_1 - \tfrac{1}{2}\rangle = +\sqrt{\frac{2j_1}{2j_1+1}}\, |m_1 = j_1, m_2 = -\tfrac{1}{2}\rangle - \sqrt{\frac{1}{2j_1+1}}\, |m_1 = j_1 - 1, m_2 = \tfrac{1}{2}\rangle}$$

How do we find lower m_j values?

Use the lowering operator again.

$$|j = j_1 + \tfrac{1}{2}, m_j = j_1 - \tfrac{3}{2}\rangle = \frac{J^-}{\hbar\sqrt{2j_1 \cdot 2}} |j = j_1 + \tfrac{1}{2}, m_j = j_1 - \tfrac{1}{2}\rangle$$

$$= \frac{1}{\sqrt{2j_1 \cdot 2(2j_1+1)}} \cdot \sqrt{2j_1 \cdot 1}\ |m_1 = j_1 - 1\ m_2 = -\tfrac{1}{2}\rangle$$

$$+ \frac{1}{\sqrt{2j_1 \cdot 2(2j_1+1)}} \sqrt{2j_1}\ \sqrt{(2j_1-1)2}\, |m_1 = j_1 - 2, m_2 = \tfrac{1}{2}\rangle$$

$$+ \frac{1}{\sqrt{2j_1 \cdot 2(2j_1+1)}} \sqrt{2j_1}\ |m_1 = j_1 - 1, m_2 = -\tfrac{1}{2}\rangle$$

$$\boxed{|j = j_1 + \tfrac{1}{2}, m_j = j_1 - \tfrac{3}{2}\rangle = \sqrt{\frac{2}{2j_1+1}}\ |m_1 = j_1 - 1, m_2 = -\tfrac{1}{2}\rangle + \sqrt{\frac{2j_1 - 1}{2j_1+1}}\ |m_1 = j_1 - 2, m_2 = +\tfrac{1}{2}\rangle}$$

In general, one finds

$$\boxed{\begin{aligned} |j = j_1 + \tfrac{1}{2}, m_j\rangle &= \sqrt{\frac{j_1 - m_j + \frac{1}{2}}{2j_1+1}}\ |m_1 = m_j + \tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle \\ &+ \sqrt{\frac{j_1 + m_j + \frac{1}{2}}{2j_1+1}}\ |m_1 = m_j - \tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle \end{aligned}}$$

Similarly

$$\boxed{|j = j_1 - \tfrac{1}{2}, m_j\rangle = \sqrt{\frac{j_1 + m_j + \frac{1}{2}}{2j_1+1}}\ |m_1 = m_j + \tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle - \sqrt{\frac{j_1 - m_j + \frac{1}{2}}{2j_1+1}}\ |m_1 = m_j - \tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle}$$

Another way to derive this is as follows:

$$|j, m_j\rangle = \alpha \, |m_1 = m_j - \tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle + \beta \, |m_1 = m_j + \tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle$$

$$\hat{J}^2 = (\hat{J}_1 + \hat{J}_2)^2 = \hat{J}_1^2 + \hat{J}_2^2 + 2\hat{J}_1 \cdot \hat{J}_2$$

$$= \hat{J}_1^2 + \hat{J}_2^2 + 2\left(\frac{J_1^+ J_2^- + J_1^- J_2^+}{2}\right) + 2 J_1^z J_2^z$$

so $\cancel{\frac{\hat{J}^2}{\hbar^2}|j, m_j\rangle = \alpha \left[j_1(j_1+1) + \frac{3}{4} + \sqrt{(j_1 - m_j + \tfrac{1}{2})(j_1 + m_j - \tfrac{1}{2} + 1)} \, |m_1 = m_j + \tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle \right.}$

$\cancel{+ 2(m_j - \tfrac{1}{2})\tfrac{1}{2}}$

$$\frac{\hat{J}^2}{\hbar^2}|j, m_j\rangle = \alpha \, |m_1 = m_j - \tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle \left(j_1(j_1+1) + \tfrac{3}{4} + 2(m_j - \tfrac{1}{2}) \cdot \tfrac{1}{2} \right) |m_1 = m_j - \tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle$$

$$+ \alpha \, |m_1 = m_j + \tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle \left(\sqrt{(j_1 - m_j + \tfrac{1}{2})(j_1 + m_j - \tfrac{1}{2} + 1)} \right)$$

$$+ \beta \, |m_1 = m_j + \tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle \left(j_1(j_1+1) + \tfrac{3}{4} + 2(m_j + \tfrac{1}{2})(-\tfrac{1}{2}) \right)$$

$$+ \beta \, |m_1 = m_j - \tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle \left(\sqrt{(j_1 + m_j + \tfrac{1}{2})(j_1 - m_j - \tfrac{1}{2} + 1)} \right)$$

$$= \begin{pmatrix} j_1(j_1+1) + m_j + \frac{1}{4} & \sqrt{(j_1 - m_j + \tfrac{1}{2})(j_1 + m_j + \tfrac{1}{2})} \\ \sqrt{(j_1 + m_j + \tfrac{1}{2})(j_1 - m_j + \tfrac{1}{2})} & j_1(j_1+1) - m_j + \frac{1}{4} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

want this to be proportional to $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ to be an eigenstate of $\hat{J}^2$.

so $$\det \begin{pmatrix} (j_1+\tfrac{1}{2})^2+m_j-\lambda & \sqrt{(j_1+\tfrac{1}{2})^2-m_j^2} \\ \sqrt{(j_1+\tfrac{1}{2})^2-m_j^2} & (j_1+\tfrac{1}{2})^2-m_j-\lambda \end{pmatrix} = 0$$

$$\Rightarrow \quad \lambda^2+\lambda\left(-2(j_1+\tfrac{1}{2})^2\right)+(j_1+\tfrac{1}{2})^4-\cancel{m_j^2}-(j_1+\tfrac{1}{2})^2+\cancel{m_j^2}=0$$

$$\lambda^2-2(j_1+\tfrac{1}{2})^2\lambda+(j_1+\tfrac{1}{2})^2\left(1+(j_1+\tfrac{1}{2})^2\right)=0$$

$$\lambda=(j_1+\tfrac{1}{2})^2\pm\tfrac{1}{2}\sqrt{4(j_1+\tfrac{1}{2})^4-4(j_1+\tfrac{1}{2})^4+4(j_1+\tfrac{1}{2})^2}$$

$$=(j_1+\tfrac{1}{2})^2\pm(j_1+\tfrac{1}{2})$$

$$=(j_1+\tfrac{1}{2})(j_1+\tfrac{1}{2}\pm 1)$$

$\Rightarrow$ one root is $j=j_1+\tfrac{1}{2}$ one is $j_1-\tfrac{1}{2}$

need to find α & β for $j=j_1\pm\tfrac{1}{2}$

$$\lambda=(j_1+\tfrac{1}{2})(j_1+\tfrac{1}{2}\pm 1)$$

$$\left((j_1+\tfrac{1}{2})^2-(j_1+\tfrac{1}{2})(j_1+\tfrac{1}{2}\pm 1)+m_j\right)\alpha+\sqrt{(j_1+\tfrac{1}{2})^2-m^2}\,\beta=0$$

$$\left(\mp(j_1+\tfrac{1}{2})+m_j\right)\alpha+\sqrt{(j_1+\tfrac{1}{2})^2-m^2}\,\beta=0$$

$$\beta=\frac{\pm(j_1+\tfrac{1}{2})-m_j}{\sqrt{(j_1+\tfrac{1}{2})^2-m_j^2}}=\pm\sqrt{\frac{j_1\mp m_j+\tfrac{1}{2}}{j_1\pm m_j+\tfrac{1}{2}}}\;\alpha$$

$$\alpha=\pm C\sqrt{j_1\pm m_j+\tfrac{1}{2}} \qquad \beta=C\sqrt{j_1\mp m_j+\tfrac{1}{2}}$$

$$\alpha^2+\beta^2=1 \quad\Rightarrow$$

$$C^2\left(j_1\pm m_j+\tfrac{1}{2}+j_1\mp m_j+\tfrac{1}{2}\right)=1 \;\Rightarrow\; C=\frac{1}{\sqrt{2j_1+1}}$$

So we get the following table

	$m_2 = \frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j = j_1 + \frac{1}{2}$	$\sqrt{\dfrac{j_1 + m_j + \frac{1}{2}}{2j_1+1}}$	$\sqrt{\dfrac{j_1 - m_j + \frac{1}{2}}{2j_1+1}}$
$j = j_1 - \frac{1}{2}$	$-\sqrt{\dfrac{j_1 - m_j + \frac{1}{2}}{2j_1+1}}$	$\sqrt{\dfrac{j_1 + m_j + \frac{1}{2}}{2j_1+1}}$

$\cancel{\Rightarrow |j, m_j\rangle}$

$$\boxed{\begin{aligned}
\Rightarrow |j_1+\tfrac{1}{2}, m_j\rangle &= \sqrt{\frac{j_1+m_j+\frac{1}{2}}{2j_1+1}}\, |m_1 = m_j - \tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle \\
&\quad + \sqrt{\frac{j_1-m_j+\frac{1}{2}}{2j_1+1}}\, |m_1 = m_j + \tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle \\
|j_1-\tfrac{1}{2}, m_j\rangle &= -\sqrt{\frac{j_1-m_j+\frac{1}{2}}{2j_1+1}}\, |m_1 = m_j - \tfrac{1}{2}, m_2 = \tfrac{1}{2}\rangle \\
&\quad + \sqrt{\frac{j_1+m_j+\frac{1}{2}}{2j_1+1}}\, |m_1 = m_j + \tfrac{1}{2}, m_2 = -\tfrac{1}{2}\rangle
\end{aligned}}$$