Phys 506 lecture 9 Central forces

This is a somewhat technical lecture.

Consider two particles interacting via a potential that depends only on the distance between them

$$\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + V(|\hat{\vec{r}}_1 - \hat{\vec{r}}_2|)$$

It turns out this problem can be mapped to an effective one-particle problem. We do so by "average" and "relative" coordinates.

The center of mass $\hat{\vec{R}}$ is defined via

$$\hat{\vec{R}} = \frac{m_1 \hat{\vec{r}}_1 + m_2 \hat{\vec{r}}_2}{m_1+m_2}$$

Because the momentum is proportional to each mass we try the average momentum to be conjugate to the cm coordinate

$$\hat{\vec{P}} = \hat{\vec{p}}_1 + \hat{\vec{p}}_2$$

check:
$$[\hat{\vec{R}}_\alpha, \hat{\vec{P}}_\beta] = \left[\frac{m_1 \hat{\vec{r}}_{1\alpha} + m_2 \hat{\vec{r}}_{2\alpha}}{m_1+m_2}, \hat{\vec{p}}_{1\beta} + \hat{\vec{p}}_{2\beta}\right]$$
$$= \frac{1}{m_1+m_2}\left(m_1[\hat{\vec{r}}_{1\alpha}, \hat{\vec{p}}_{1\beta}] + m_2[\hat{\vec{r}}_{2\alpha}, \hat{\vec{p}}_{2\beta}]\right)$$
$$= \frac{1}{m_1+m_2}(m_1 i\hbar \delta_{\alpha\beta} + m_2 i\hbar\delta_{\alpha\beta})$$
$$= i\hbar\delta_{\alpha\beta} \ \checkmark$$

Now examine the relative position

$$\hat{\vec{r}} = \hat{\vec{r}}_1 - \hat{\vec{r}}_2$$

Since momentum is proportional to mass, try

$$\hat{\vec{p}}_{rel} = \frac{m_1 m_2}{m_1+m_2}\left(\frac{\hat{\vec{p}}_1}{m_1} - \frac{\hat{\vec{p}}_2}{m_2}\right)$$

check:
$$[\hat{\vec{r}}_\alpha, \hat{\vec{p}}_{rel\,\beta}] = \left[\hat{\vec{r}}_{1\alpha} - \hat{\vec{r}}_{2\alpha}, \frac{m_1 m_2}{(m_1+m_2)}\left(\frac{\hat{\vec{p}}_{1\beta}}{m_1} - \frac{\hat{\vec{p}}_{2\beta}}{m_2}\right)\right]$$
$$= \frac{m_1 m_2}{(m_1+m_2)}\left(\left[\hat{r}_{1\alpha}, \frac{\hat{p}_{1\beta}}{m_1}\right] - \left[\hat{r}_{2\alpha}, -\frac{\hat{p}_{2\beta}}{m_2}\right]\right)$$
$$= \frac{m_1 m_2}{(m_1+m_2)}\left(\frac{1}{m_1} i\hbar\delta_{\alpha\beta} + \frac{1}{m_2} i\hbar\delta_{\alpha\beta}\right)$$
$$= i\hbar\delta_{\alpha\beta}$$

Furthermore $[\hat{\vec{R}}_\alpha, \hat{\vec{p}}_{rel\,\beta}] = [\hat{\vec{r}}_\alpha, \hat{\vec{P}}_\beta] = 0$, which you can easily check.

Note further that

$$\hat{\vec{p}}_1 = \left(\hat{\vec{p}} + \frac{m_1+m_2}{m_1}\hat{\vec{p}}_{rel}\right)\frac{m_1}{m_1+m_2}$$
$$\hat{\vec{p}}_2 = \left(\hat{\vec{p}} - \frac{m_1+m_2}{m_2}\hat{\vec{p}}_{rel}\right)\frac{m_2}{m_1+m_2}$$

so
$$\frac{\hat{\vec{p}}_1^{\,2}}{2m_1} + \frac{\hat{\vec{p}}_2^{\,2}}{2m_2} = \frac{1}{2}\frac{m_1}{(m_1+m_2)^2}\left(\hat{\vec{p}}^{\,2} + 2\frac{m_1+m_2}{m_1}\hat{\vec{p}}\cdot\hat{\vec{p}}_{rel} + \left(\frac{m_1+m_2}{m_1}\right)^2\hat{\vec{p}}_{rel}^{\,2}\right)$$
$$+ \frac{1}{2}\frac{m_2}{(m_1+m_2)^2}\left(\hat{\vec{p}}^{\,2} - 2\frac{m_1+m_2}{m_2}\hat{\vec{p}}\cdot\hat{\vec{p}}_{rel} + \left(\frac{m_1+m_2}{m_2}\right)^2\hat{\vec{p}}_{rel}^{\,2}\right)$$
$$= \frac{1}{2}\frac{\hat{\vec{p}}^{\,2}}{(m_1+m_2)} + \frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\hat{\vec{p}}_{rel}^{\,2}$$

define $\mu = \frac{m_1 m_2}{m_1+m_2}$ = reduced mass

$$\frac{\hat{\vec{p}}_1^{\,2}}{2m_1} + \frac{\hat{\vec{p}}_2^{\,2}}{2m_2} = \frac{1}{2}\frac{\hat{\vec{p}}^{\,2}}{(m_1+m_2)} + \frac{1}{2}\frac{\hat{\vec{p}}_{rel}^{\,2}}{\mu}$$

and $\hat{H}$ becomes
$$\underbrace{\frac{1}{2}\frac{\hat{\vec{p}}^{\,2}}{(m_1+m_2)}}_{\substack{\text{CM motion}\\ \text{free particle}}} + \underbrace{\frac{1}{2}\frac{\hat{\vec{p}}_{rel}^{\,2}}{\mu} + V(|\hat{\vec{r}}|)}_{\substack{\text{relative motion}\\ \text{effective particle in potential}}}$$

We can separate the motion into CM and relative motion. Only change is we use the reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$.

We drop the rel subscript now and want to rewrite the kinetic energy in terms of the radial and angular contributions

We start with the radial momentum
$$\hat{p}_r = \frac{1}{2}\left(\frac{\hat{\vec{r}}}{\hat{r}}\cdot\hat{\vec{p}} + \hat{\vec{p}}\cdot\frac{\hat{\vec{r}}}{\hat{r}}\right)$$
(we use a symmetric form because it is hermitian)

on HW, you will show
$$[\hat{p}_\alpha, \hat{r}] = -i\hbar\frac{\hat{r}_\alpha}{\hat{r}}$$

and $$\left[\hat{p}_\alpha, \frac{1}{\hat{r}}\right] = i\hbar\frac{\hat{r}_\alpha}{\hat{r}^3}$$

$$\Rightarrow \hat{p}_r = \frac{1}{2}\left(\frac{\hat{\vec{r}}}{\hat{r}}\cdot\hat{\vec{p}} + \frac{\hat{\vec{r}}}{\hat{r}}\cdot\hat{\vec{p}} + \sum_\alpha\left[\hat{p}_\alpha, \frac{\hat{r}_\alpha}{\hat{r}}\right]\right)$$
$$= \frac{1}{\hat{r}}\hat{\vec{r}}\cdot\hat{\vec{p}} + \frac{1}{2}\left(-i\hbar\times3\times\frac{1}{\hat{r}} + \sum_\alpha \hat{r}_\alpha\, i\hbar\frac{\hat{r}_\alpha}{\hat{r}^3}\right)$$

$$\hat{p}_r = \frac{1}{\hat{r}}\hat{\vec{r}}\cdot\hat{\vec{p}} - \frac{i\hbar}{\hat{r}}$$ = radial momentum.

$$[\hat{r}, \hat{p}_r] = [\hat{r}, \tfrac{1}{\hat{r}} \hat{\vec{r}}\cdot\hat{\vec{p}} - \tfrac{i\hbar}{\hat{r}}]$$
$$= \frac{1}{\hat{r}} \sum_\alpha \hat{r}_\alpha [\hat{r}, \hat{p}_\alpha]$$
$$= \frac{1}{\hat{r}} \sum_\alpha \hat{r}_\alpha \, i\hbar \frac{\hat{r}_\alpha}{\hat{r}} = i\hbar$$

So we think of $\hat{r}$ and $\hat{p}_r$ as canonically conjugate operators.
Note that $\hat{p}_r$ is Hermitian, but is not self-adjoint, so it has no set of eigenstates. This does not affect anything we do.

Compute $\hat{p}_r^2 = \frac{1}{\hat{r}} (\hat{\vec{r}}\cdot\hat{\vec{p}} - i\hbar) \frac{1}{\hat{r}} (\hat{\vec{r}}\cdot\hat{\vec{p}} - i\hbar)$

since $[\hat{r}, \hat{p}_r] = i\hbar \Rightarrow [\frac{\hat{r}}{\hat{r}}, \hat{p}_r] = \frac{1}{\hat{r}}[\hat{r}, \hat{p}_r] + [\frac{1}{\hat{r}}, \hat{p}_r]\hat{r}$
$$= 0$$
$$\Rightarrow [\tfrac{1}{\hat{r}}, \hat{p}_r] = -\frac{i\hbar}{\hat{r}^2}$$

so $\hat{p}_r^2 = \left(\frac{1}{\hat{r}^2} (\hat{\vec{r}}\cdot\hat{\vec{p}} - i\hbar) + [\hat{p}_r, \tfrac{1}{\hat{r}}]\right)(\hat{\vec{r}}\cdot\hat{\vec{p}} - i\hbar)$
$$= \frac{1}{\hat{r}^2} \hat{\vec{r}}\cdot\hat{\vec{p}} (\hat{\vec{r}}\cdot\hat{\vec{p}} - i\hbar)$$
$$= \frac{1}{\hat{r}^2} \sum_{\alpha\beta} \hat{r}_\alpha \hat{p}_\alpha (\hat{r}_\beta \hat{p}_\beta - i\hbar)$$
$$= \frac{1}{\hat{r}^2} \sum_{\alpha\beta} \left(\hat{r}_\alpha \hat{r}_\beta \hat{p}_\alpha \hat{p}_\beta - \hat{r}_\alpha \hat{p}_\beta i\hbar \delta_{\alpha\beta} - i\hbar \hat{r}_\alpha \hat{p}_\alpha\right)$$
$$= \frac{1}{\hat{r}^2}\left(\sum_{\alpha\beta} \hat{r}_\alpha \hat{r}_\beta \hat{p}_\alpha \hat{p}_\beta - 2i\hbar \hat{\vec{r}}\cdot\hat{\vec{p}}\right)$$
$$= \frac{1}{\hat{r}^2}\Big[\hat{r}_x^2 \hat{p}_x^2 + \hat{r}_y^2 \hat{p}_y^2 + \hat{r}_z^2 \hat{p}_z^2 + 2\hat{r}_x\hat{r}_y\hat{p}_x\hat{p}_y + 2\hat{r}_x\hat{r}_z\hat{p}_x\hat{p}_z + 2\hat{r}_y\hat{r}_z\hat{p}_y\hat{p}_z - 2i\hbar(\hat{r}_x\hat{p}_x + \hat{r}_y\hat{p}_y + \hat{r}_z\hat{p}_z)\Big]$$

Similarly $(\hat{\vec{r}}\times\hat{\vec{p}})^2 = (\hat{r}_x\hat{p}_y - \hat{r}_y\hat{p}_x)(\hat{r}_x\hat{p}_y - \hat{r}_y\hat{p}_x) + (\hat{r}_y\hat{p}_z - \hat{r}_z\hat{p}_y)(\hat{r}_y\hat{p}_z - \hat{r}_z\hat{p}_y) + (\hat{r}_z\hat{p}_x - \hat{r}_x\hat{p}_z)(\hat{r}_z\hat{p}_x - \hat{r}_x\hat{p}_z)$

$$= \hat{r}_x^2\hat{p}_y^2 + \hat{r}_y^2\hat{p}_x^2 - 2\hat{r}_x\hat{r}_y\hat{p}_x\hat{p}_y + i\hbar\hat{r}_y\hat{p}_y + i\hbar\hat{r}_x\hat{p}_x$$
$$+ \hat{r}_y^2\hat{p}_z^2 + \hat{r}_z^2\hat{p}_y^2 - 2\hat{r}_z\hat{r}_y\hat{p}_y\hat{p}_z + i\hbar\hat{r}_z\hat{p}_z + i\hbar\hat{r}_y\hat{p}_y$$
$$+ \hat{r}_z^2\hat{p}_x^2 + \hat{r}_x^2\hat{p}_z^2 - 2\hat{r}_x\hat{r}_z\hat{p}_x\hat{p}_z + i\hbar\hat{r}_x\hat{p}_x + i\hbar\hat{r}_z\hat{p}_z$$

$$\Rightarrow \quad \hat{p}_r^2 + \frac{1}{\hat{r}^2}\hat{L}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$$

so
$$\boxed{\hat{H} = \frac{\hat{p}_r^2}{2\mu} + \frac{\hat{L}^2}{2\mu\hat{r}^2} + V(\hat{r})}$$

The last thing we work on is the radial translation operator. It is just the translation operator expressed in terms of $\hat{p}_r$, $\hat{p}_\theta$, $\hat{p}_\phi$, $\cos\hat{\theta}$, $\sin\hat{\theta}$, $\cos\hat{\phi}$, and $\sin\hat{\phi}$. Doing so requires a few subtle points. We do not go through the full details, but do describe most of the issues.

$$e^{-\frac{i}{\hbar}(x\hat{p}_x + y\hat{p}_y + z\hat{p}_z)}$$

Recall $\hat{p}_r = \frac{1}{\hat{r}}\left(\hat{r}_x \hat{p}_x + \hat{r}_y \hat{p}_y + \hat{r}_z \hat{p}_z - i\hbar\right)$

$$= \sin\hat{\theta}\cos\hat{\phi}\,\hat{p}_x + \sin\hat{\theta}\sin\hat{\phi}\,\hat{p}_y + \cos\hat{\theta}\,\hat{p}_z - \frac{i\hbar}{\hat{r}}$$

We also compute the analogs of $\hat{p}_\theta = \hat{\vec{e}}_\theta \cdot \hat{\vec{p}}$ and $\hat{p}_\phi = \hat{\vec{e}}_\phi \cdot \hat{\vec{p}}$

similar to how we computed $\hat{p}_r$

$$\hat{p}_\theta = \cos\hat{\theta}\cos\hat{\phi}\,\hat{p}_x + \cos\hat{\theta}\sin\hat{\phi}\,\hat{p}_y - \sin\hat{\theta}\,\hat{p}_z - \frac{i\hbar}{2}\frac{\cot\hat{\theta}}{\hat{r}}$$

$$\hat{p}_\phi = -\sin\hat{\phi}\,\hat{p}_x + \cos\hat{\phi}\,\hat{p}_y$$

$\hat{p}_\theta$ requires a quantum correction, but $\hat{p}_\phi$ does not. You will show this on the homework.

We use $x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$
(here it is the x,y,z of the translations, all are numbers)

Our strategy is to show that the translation can be written as a translation along z followed by the rotations by θ along y axis and ϕ along z axis.

$$|x,y,z\rangle = e^{-i\frac{\phi}{\hbar}\hat{L}_z} e^{-i\frac{\theta}{\hbar}\hat{L}_y} e^{-i\frac{r}{\hbar}\hat{p}_z} |0\rangle$$

$$= e^{-i\frac{\phi}{\hbar}\hat{L}_z} e^{-i\frac{\theta}{\hbar}\hat{L}_y} e^{-i\frac{r}{\hbar}\hat{p}_z} \underbrace{e^{i\frac{\theta}{\hbar}\hat{L}_y} e^{i\frac{\phi}{\hbar}\hat{L}_z} |0\rangle}_{|0\rangle}$$

because $\hat{L}_z = \hat{r}_x \hat{p}_y - \hat{r}_y \hat{p}_x$ and $\hat{r}_\alpha |0\rangle = 0$
$\hat{L}_y = \hat{r}_z \hat{p}_x - \hat{r}_x \hat{p}_z$

But $e^{-i\frac{\theta}{\hbar}\hat{L}_y}\,\hat{p}_z\, e^{i\frac{\theta}{\hbar}\hat{L}_y} = \sin\theta\,\hat{p}_x + \cos\theta\,\hat{p}_z$

$$e^{-i\frac{\phi}{\hbar}\hat{L}_z}\,\hat{p}_x\, e^{i\frac{\phi}{\hbar}\hat{L}_z} = \cos\phi\,\hat{p}_x + \sin\phi\,\hat{p}_y$$

$$\Rightarrow |x,y,z\rangle = e^{-i\frac{r}{\hbar}\left(\sin\theta\cos\phi\,\hat{p}_x + \sin\theta\sin\phi\,\hat{p}_y + \cos\theta\,\hat{p}_z\right)}|0\rangle$$

$$= e^{-\frac{i}{\hbar}(x\hat{p}_x + y\hat{p}_y + z\hat{p}_z)}|0\rangle$$ as expected.

Now go back to our original form

$$|x,y,z\rangle = e^{-i\frac{\phi}{\hbar}\hat{L}_z} e^{-i\frac{\theta}{\hbar}\hat{L}_y} e^{-i\frac{r}{\hbar}\hat{p}_z} |0\rangle$$

But $\hat{p}_z = \cos\hat{\theta}\,\hat{p}_r - \sin\hat{\theta}\,\hat{p}_\theta + i\hbar\frac{\cos\hat{\theta}}{\hat{r}} - i\frac{\hbar}{2\hat{r}}\cos\hat{\theta}$

This implies
$$|x,y,z\rangle = e^{-i\frac{\phi}{\hbar}\hat{L}_z}\, e^{-i\frac{\theta}{\hbar}\hat{L}_y}\, e^{-i\frac{r}{\hbar}\left(\cos\hat\theta\,\hat P_r - \sin\hat\theta\,\hat P_\theta + \frac{i\hbar}{2\hat r}\cos\hat\theta\right)}|0\rangle \qquad (5.)$$

Note that because we are translating along the z-axis, we will have $\cos\hat\theta\,|0\rangle = |0\rangle$ and $\sin\hat\theta\,|0\rangle = 0$

recall
$$\hat P_r = \frac{1}{\hat r}\left(\hat r_x \hat P_x + \hat r_y \hat P_y + \hat r_z \hat P_z\right) - \frac{i\hbar}{\hat r}$$

$$\left[\frac{\hat r_\alpha}{\hat r}, \hat P_r\right] = \frac{1}{\hat r}\left[\hat r_\alpha, \hat P_r\right] + \left[\frac{1}{\hat r}, \hat P_r\right]\hat r_\alpha$$
$$= \frac{1}{\hat r^2}\hat r_\alpha\, i\hbar + -i\hbar\,\frac{1}{\hat r^2}\hat r_\alpha = 0$$
$\Rightarrow$ $\hat P_r$ commutes with $\cos\hat\theta$ and $\sin\hat\theta$.

$$\hat P_\theta = \cos\hat\theta\cos\hat\phi\,\hat P_x + \cos\hat\theta\sin\hat\phi\,\hat P_y - \sin\hat\theta\,\hat P_z - \frac{i\hbar}{2}\frac{\cot\hat\theta}{\hat r}$$

Note that
$$-\sin\hat\phi\,\hat L_x + \cos\hat\phi\,\hat L_y = -\sin\hat\phi\sin\hat\theta\sin\hat\phi\,\hat r\,\hat P_z + \sin\hat\phi\cos\hat\theta\,\hat r\,\hat P_y$$
$$+\cos\hat\phi\cos\hat\theta\,\hat r\,\hat P_x - \cos\hat\phi\sin\hat\theta\cos\hat\phi\,\hat r\,\hat P_z$$
$$= \cos\hat\theta\cos\hat\phi\,\hat r\,\hat P_x + \cos\hat\theta\sin\hat\phi\,\hat r\,\hat P_y - \sin\hat\theta\,\hat r\,\hat P_z$$

But
$$\sin\hat\theta\,\hat P_\theta = \sin\hat\theta\cos\hat\theta\cos\hat\phi\,\hat P_x + \sin\hat\theta\cos\hat\theta\sin\hat\phi\,\hat P_y - \sin^2\hat\theta\,\hat P_z - \frac{i\hbar}{2\hat r}\cos\hat\theta$$
$$= \frac{\sin\hat\theta}{\hat r}\left(-\sin\hat\phi\,\hat L_x + \cos\hat\phi\,\hat L_y\right) - \frac{i\hbar}{2\hat r}\cos\hat\theta$$
$$= \frac{-\hat r_y\hat L_x + \hat r_x\hat L_y}{\hat r^2} - \frac{i\hbar}{2\hat r}\cos\hat\theta$$

Hence
$$\cos\hat\theta\,\hat P_r - \sin\hat\theta\,\hat P_\theta + \frac{i\hbar}{2\hat r}\cos\hat\theta$$
$$= \cos\hat\theta\,\hat P_r + \frac{\hat r_y\hat L_x - \hat r_x\hat L_y}{\hat r^2} + \frac{i\hbar}{\hat r}\cos\hat\theta$$
$$= \left(\hat P_r + \frac{i\hbar}{\hat r}\right)\cos\hat\theta + \frac{\hat r_y\hat L_x - \hat r_x\hat L_y}{\hat r^2}$$
$$= \left(\hat P_r + \frac{i\hbar}{\hat r}\right)\cos\hat\theta + \frac{\hat L_x\hat r_y - \hat L_y\hat r_x}{\hat r^2} - i\hbar\frac{\hat r_z}{\hat r^2} - i\hbar\frac{\hat r_z}{\hat r^2}$$
$$= \left(\hat P_r - \frac{i\hbar}{\hat r}\right)\cos\hat\theta + \frac{\hat L_x\sin\hat\theta\sin\hat\phi - \hat L_y\sin\hat\theta\cos\hat\phi}{\hat r}$$
$$= \left(\hat P_r - \frac{i\hbar}{\hat r}\right)\cos\hat\theta + \hat P_z\sin^2\hat\theta\sin^2\hat\phi - \hat P_y\cos\hat\theta\sin\hat\theta\sin\hat\phi$$
$$- \hat P_x\cos\hat\theta\sin\hat\theta\cos\hat\phi + \hat P_z\sin^2\hat\theta\cos^2\hat\phi$$

$$= \left(\hat P_r - \frac{i\hbar}{\hat r}\right)\cos\hat\theta - \hat P_x\cos\hat\theta\sin\hat\theta\cos\hat\phi - \hat P_y\cos\hat\theta\sin\hat\theta\sin\hat\phi + \hat P_z\sin^2\hat\theta$$

when this acts on $|0\rangle$ it gives $\left(\hat P_r - \frac{i\hbar}{\hat r}\right)|0\rangle$
because $\sin\hat\theta\,|0\rangle = 0$

Furthermore, since $\sin\hat{\theta}$ commutes with $\hat{p}_r - \frac{i\hbar}{\hat{r}}$, we find raising this operator to any power acting on $|0\rangle$ also gives $(\hat{p}_r - \frac{i\hbar}{\hat{r}})$ raised to that power acting on $|0\rangle$

This gives our final result.

$$\boxed{|x,y,z\rangle = e^{-i\frac{\phi}{\hbar}\hat{L}_z}\, e^{-i\frac{\theta}{\hbar}\hat{L}_y}\, e^{-i\frac{r}{\hbar}\left(\hat{p}_r - \frac{i\hbar}{\hat{r}}\right)}|0\rangle}$$

This is the translation operator in spherical coordinates.

Note that it has a term that appears to be non unitary, but as we saw, this operator does produce the correct position space state. Indeed, the quantum correction term is critical to it giving the right behavior.