Physics 515 lecture 1

Cauchy's Theorem and the calculus of residues

We will study functions of a complex variable

$$z = x + iy \qquad x, y \text{ real} \qquad i = \sqrt{-1}$$

analytic functions are functions of z with special properties, one being that they are infinitely differentiable.

examples of analytic functions e^z, $\cos z$, $\sin z$

some analytic functions need care in defining their domain that they live on like $\sqrt{z}$ or $\ln z$.

We will take as an ansatz, or if you like as a definition, that every analytic function has an antiderivative, so if $f(z)$ is analytic, there exists a $g(z)$ such that

$$f(z) = \frac{d}{dz} g(z)$$

(note, this is not the mathematical definition of an analytic function)

Now consider a path γ that runs from $\gamma(a) = z_1$ to $\gamma(b) = z_2$ (γ is parametrized by a real variable t).

the integral, called a contour integral, is denoted

$$\int_\gamma f\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

but since f has an antiderivative, we write

$$\int_\gamma f\,dz = \int_\gamma \frac{d}{dz} g\,dz = \int_\gamma dg = g(\gamma(b)) - g(\gamma(a))$$

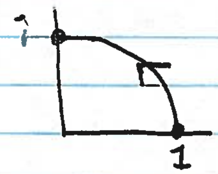
For a closed path $z_1 = z_2$ or $\gamma(a) = \gamma(b)$ and we get

$$\int_\gamma f\,dz = 0 \qquad \gamma = \text{closed path}$$

which is a form of Cauchy's theorem.

The integral is independent of the shape of γ, which means we can deform the contour to different shapes.

example: Integrate e^z along the unit circle for one quadrant from 1 to i



parametrize $\gamma(t) = e^{it} \qquad 0 \le t \le \frac{\pi}{2}$

$$\gamma'(t) = i e^{it}$$

so $\displaystyle\int_\gamma e^z\,dz = \int_0^{\pi/2} e^{\cos t + i\sin t}\,(i\cos t - \sin t)\,dt$

$$= \int_0^{\pi/2} e^{\cos t}\left(\cos(\sin t) + i\sin(\sin t)\right)(-\sin t + i\cos t)\,dt$$

$$= \int_0^{\pi/2} e^{\cos t}\left(-\cos(\sin t)\sin t - \sin(\sin t)\cos t\right)dt$$

$$+ i\int_0^{\pi/2} e^{\cos t}\left(-\sin(\sin t)\sin t + \cos(\sin t)\cos t\right)dt$$

$$= \int_0^{\pi/2} \frac{d}{dt}\left(e^{\cos t}\cos(\sin t)\right)dt + i\int_0^{\pi/2} \frac{d}{dt}\left(e^{\cos t}\sin(\sin t)\right)dt$$

$$= e^{\cos t}\cos(\sin t)\Big|_0^{\pi/2} + i\, e^{\cos t}\sin(\sin t)\Big|_0^{\pi/2} =$$

$$= e^{\cos t + i\sin t}\Big|_0^{\pi/2} = e^{i} - e^{1}$$

The alternative way is to note

$$g(z) = e^z$$

so $\int_\gamma f\,dz = g(z_2) - g(z_1) = e^i - e^1$

which is much easier! It also depends only on the endpoints.

Let γ be a circle centred around a, in the counterclockwise direction

compute $\int_\gamma (z-a)^n\,dz$

$$g(z) = \frac{1}{n+1}(z-a)^{n+1} \quad \text{except } n = -1$$

so for all n except $n=-1$, the integral is zero.

Let $\gamma = a + re^{it}$ $\quad 0 \le t \le 2\pi$ $\quad \gamma' = re^{it}\,i$

$$\int_\gamma (z-a)^{-1}\,dz = \int_0^{2\pi} \frac{1}{r} e^{-it}\, i r e^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$$

so we find $\int_\gamma (z-a)^n\,dz = \begin{cases} 0 & n \ne -1 \\ 2\pi i & n = -1 \end{cases}$

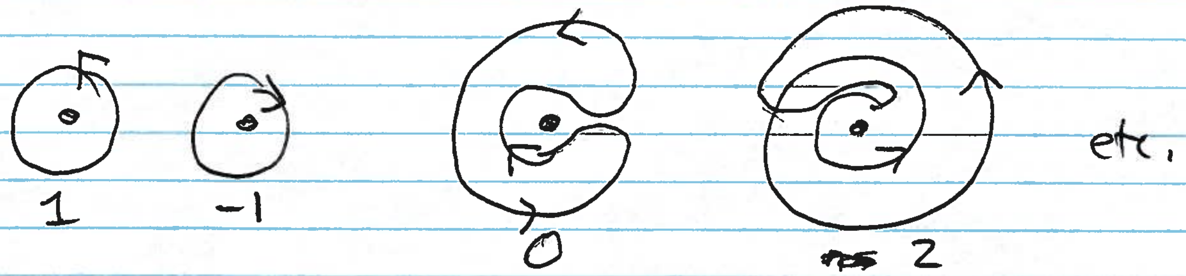
Note: we could not use the logarithm to define $g(z)$ for $n=-1$ because the complex logarithm increases by $2\pi i$ as we go around the circle, so it is not continuous hence not analytic around the entire circle.

The precise definition of Cauchy's theorem does not require the function to have an antiderivative, just to be analytic in the interior of the closed contour then

$\oint_\gamma f\,dz = 0$ $\quad$ analytic implies no divergences inside the contour area and infinitely differentiable.

once again, the curve can be deformed as long as f is analytic in the region.

Define the <u>index</u> of a <u>path</u> about a point z_0 to be the number of times it winds around z_0.



denote the index by $I(\gamma, z_0)$. Then Cauchy's theorem is

$$f(z_0)\, I(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\, dz}{z - z_0}$$

or, in most cases we take curves with index $= 1$.

Then

$$\boxed{f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz}$$

z_0 = inside contour γ
γ winds 1 time around z_0 (ccw)

The function on the interior is determined entirely by all values on the boundary! This is an amazing property of analytic functions.

You might want to compare to something like Gauss' law where a volume integral is equal to an integral over a surface, etc. Though different, the ideas are similar.

We now consider a generalization of the Taylor series expansion called a Laurent expansion

$$f(z) = \cdots \frac{b_n}{(z-z_0)^n} + \frac{b_{n-1}}{(z-z_0)^{n-1}} + \cdots \frac{b_1}{z-z_0} + \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}_{\text{Taylor series expansion}}$$

The coefficient b_1 is called the residue of f at z_0.

If all $b_{n \geq 2} = 0$ f has a simple pole at z_0

If all $b_{n \geq 3} = 0$ f has a double pole at z_0

Example:

suppose $h(z_0) = 0$ $h'(z_0) \neq 0$ $g(z_0) \neq 0$, is finite

let $f(z) = \dfrac{g(z)}{h(z)}$ (simple pole)

But
$$g(z) = g(z_0) + (z-z_0)\,g'(z_0) + \tfrac{1}{2}(z-z_0)^2 g''(z_0) + \cdots$$
$$h(z) = (z-z_0)\,h'(z_0) + \tfrac{1}{2}(z-z_0)^2 h''(z_0) + \cdots$$

$$f(z) = \frac{\dfrac{g(z_0)}{z-z_0} + g'(z_0) + \tfrac{1}{2}(z-z_0)\,g''(z_0)}{h'(z_0) + \tfrac{1}{2}(z-z_0)\,h''(z_0) + \cdots}$$

$$= \left[\frac{g(z_0)}{(z-z_0)h'(z_0)} + \frac{g'(z_0)}{h'(z_0)} + \tfrac{1}{2}(z-z_0)\,g''(z_0)\right]\left[1 - \tfrac{1}{2}(z-z_0)\frac{h''(z_0)}{h'(z_0)} + \cdots\right]$$

$$= \frac{g(z_0)}{(z-z_0)h'(z_0)} + \frac{g'(z_0)}{h'(z_0)} - \frac{1}{2}\frac{g(z_0)\,h''(z_0)}{(h'(z_0))^2} + \cdots$$

near z_0. $\Rightarrow$ residue $= g(z_0) / h'(z_0)$ (simple pole) let $g(z_0) \neq 0$, finite

$h(z_0) = h'(z_0) = 0 \quad h''(z_0) \neq 0$ (double pole)

$$f(z) = \frac{g(z)}{h(z)} = \frac{g(z_0) + (z-z_0)\, g'(z_0) + \frac{1}{2}(z-z_0)^2 g''(z_0) + \cdots}{\frac{1}{2}(z-z_0)^2 h''(z_0) + \frac{1}{6}(z-z_0)^3 h'''(z_0) + \cdots}$$

$$= \frac{g(z_0) + (z-z_0)\, g'(z_0) + \frac{1}{2}(z-z_0)^2 g''(z_0) + \cdots}{\frac{1}{2}(z-z_0)^2 h''(z_0)\left(1 + \frac{1}{3}(z-z_0)\frac{h'''(z_0)}{h''(z_0)} + \cdots\right)}$$

$\uparrow$ small near z_0

$$= \left[\frac{2g(z_0)}{(z-z_0)^2 h''(z_0)} + \frac{2g'(z_0)}{(z-z_0)\, h''(z_0)} + \frac{g''(z_0)}{h''(z_0)} + \cdots\right]$$

$$* \left[1 - \frac{1}{3}(z-z_0)\frac{h'''(z_0)}{h''(z_0)} + \cdots\right]$$

$$= \frac{2g(z_0)}{(z-z_0)^2 h''(z_0)} + \frac{1}{z-z_0}\left[\frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)\, h'''(z_0)}{[h''(z_0)]^2}\right] + \cdots$$

so residue $= \dfrac{2g'(z_0)}{h''(z_0)} - \dfrac{2}{3}\dfrac{g(z_0)\, h'''(z_0)}{[h''(z_0)]^2}$

higher order poles are more complicated to calculate.

one final example:

residue of $\dfrac{g(z)}{(z-z_0)^n} = \dfrac{1}{(n-1)!}\, g^{(n-1)}(z_0)$

since $g(z) = g(z_0) + (z-z_0)\, g'(z_0) + \cdots \frac{1}{(n-1)!}(z-z_0)^{n-1} g^{(n-1)}(z_0) + \cdots$

The residue theorem

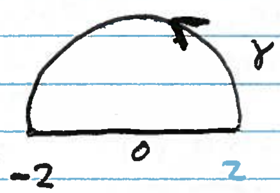
let γ be a curve that goes around a number of poles of a function f.

$$\int_\gamma f\,dz = \sum_{j=1}^{n} 2\pi i \ \mathrm{Res}(f, z_j)\, I(\gamma, z_j)$$

where z_j are poles of f that γ winds around.

This theorem is primarily used to evaluate integrals.

example.



$\gamma =$ contour running from -2 to 2 on the real axis then a semicircle of radius 2 from 2 to -2.

$$\int_\gamma \frac{1}{z^4+1}\,dz$$

when does $z^4+1=0$

Fourth root of -1 $\quad e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$

$\times$ marks a pole of $f(z) = \dfrac{1}{z^4+1}$

obviously we wind with $I=+1$ around the $e^{i\pi/4}$ and $e^{3i\pi/4}$ poles.

so $$\int_\gamma \frac{1}{z^4+1}\,dz = 2\pi i \left\{ \mathrm{Res}(e^{i\pi/4}) * (+1) + \mathrm{Res}(e^{3i\pi/4}) * (+1) \right\}$$

we write $$\frac{1}{z^4+1} = \frac{1}{(z-e^{i\pi/4})(z-e^{3i\pi/4})(z-e^{5i\pi/4})(z-e^{7i\pi/4})}$$

so the residue at $e^{i\pi/4}$ is (set $z = e^{i\pi/4}$, remove $z - e^{i\pi/4}$ term)

$$\frac{1}{(e^{i\pi/4} - e^{3i\pi/4})(e^{i\pi/4} - e^{5i\pi/4})(e^{i\pi/4} - e^{7i\pi/4})}$$

and at $e^{3i\pi/4}$ is (set $z = e^{3i\pi/4}$ remove $z - e^{3i\pi/4}$ term)

$$\frac{1}{(e^{3i\pi/4} - e^{i\pi/4})(e^{3i\pi/4} - e^{5i\pi/4})(e^{3i\pi/4} - e^{7i\pi/4})}$$

using $e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$ $\quad e^{3i\pi/4} = \frac{1}{\sqrt{2}}(-1+i)$ $\quad e^{5i\pi/4} = \frac{1}{\sqrt{2}}(-1-i)$

$e^{7i\pi/4} = \frac{1}{\sqrt{2}}(+1-i)$ gives

$$\text{Res}(e^{i\pi/4}) = \frac{1}{(\sqrt{2})(\sqrt{2}(1+i))(\sqrt{2}\,i)} = \frac{1}{2\sqrt{2}}\frac{1}{-1+i} = -\frac{1}{2\sqrt{2}}\frac{1+i}{2}$$

$$= -\frac{1}{4\sqrt{2}}(1+i)$$

$$\text{Res}(e^{3i\pi/4}) = \frac{1}{(-\sqrt{2})(\sqrt{2}\,i)(\sqrt{2}(-1+i))} = \frac{1}{2\sqrt{2}}\frac{1}{1+i} = \frac{1}{4\sqrt{2}}(1-i)$$

$$\text{so integral} = \frac{2\pi i}{4\sqrt{2}}(-1-i+1-i) = \frac{\pi}{\sqrt{2}} = \int_\gamma \frac{dz}{z^4+1}$$