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Conformal Deformation by the Currents of Affine g

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We develop a quasi-systematic approach to continuous parameters in conformal and superconformal field theory. The formulation unifies continuous twists, ghosts, and mechanisms of spontaneous breakdown in a general hierarchy of conformal deformations about a given theory by its own currents. Highlights include continuously twisted Sugawara and coset constructions, generalized ghosts, classes of $N=1$ and 2 superconformal field theories with continuous central charge, vertex-operators for arbitrarily deformed lattices, operator-valued conformal weights and/or central charges, and generalizations of continuous $SO(p, q)$ families of conformal field theories.

Contents.

1. INTRODUCTION

Affine Lie algebra, or current algebra on S^1, was discovered independently in mathematics [1] and physics [2]. The first representations [2, 3] (level-one of $SU(N)$, $SO(3, 1)$) were constructed with world-sheet fermions [2, 4] to implement the proposal of current-algebraic spin and internal symmetry on the string [2].

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Many other conformal [5], affine-conformal [2], and superconformal [4, 6, 7] constructions were advanced in those days, including Sugawara's construction [8], Sugawara constructions on the string [2, 9, 10], and the coset constructions [2, 9, 11]. The vertex-operator construction of fermions and level-one of $SU(N)$ from compactified spatial dimensions [12, 13] and the generalization to level-one of simply laced g [14, 15] was among the last developments before communication between mathematicians and physicists was established in the modern era [16–18]. A parallel development in physics was the gradual understanding of dual models as strings and conformal field theory [5, 19, 12, 20–22], in which the constructions above play the role of (chiral) conformal building-blocks for modular-invariant [23] theories.

Continuous families of conformal and superconformal field theories have been a topic of recent interest as a mechanism for spontaneous breakdown on the string [24, 25]. Two conformal constructions with continuous parameters

$$L_{BH}(\tilde{a}_0^8)=\frac{i}{2}\bar{\psi}\bar{\partial}_\theta\psi+\frac{\tilde{a}_0^8\bar{\psi}\lambda_8\psi}{\sqrt{2}}+\frac{1}{2}(\tilde{a}_0^8)^2, \qquad c(\tilde{a}_0^8)=3 \tag{1.1a}$$

$$L_F(D_0)=\frac{1}{2}(\partial_\theta Q)^2+D_0 i\partial_\theta^2 Q-\frac{1}{2}D_0^2, \qquad c(D_0)=1-12D_0^2 \tag{1.1b}$$

were known in the previous era: The fermionic $SU(3)$ construction (1.1a) was given in the original paper of Bardakci and Halpern [2] as a mechanism for the continuous breaking of internal symmetry on the string, and in fact contains the first continuously twisted affine Lie algebra (see Section 4.1). The bosonic $U(1)$ construction (1.1b) given by Fairlie in [26] was the prototype ghost [27, 28] construction, since the central charge varies continuously. Both constructions are linear in the currents of the theories defined at vanishing values of their respective parameters.

The present work unifies and generalizes these prototypes in a quasi-systematic approach to continuous parameters which we call conformal deformations. The formulation employs the full machinery of affine, conformal, and superconformal systems to construct a general hierarchy of deformations about a given conformal or superconformal field theory by its own currents:

1. the *flat-deformations*, which unify continuous (inner-automorphic) twists and generalized ghosts (Sections 2–7),

2. the *linear-loaded deformations*, which generalize the flat deformations and include known mechanisms of spontaneous breakdown (Sections 2 and 8), and

3. the *arbitrarily loaded deformations*, which generalize the linear-loaded deformations and lie generically just outside the boundary of present model-building (Sections 2, 9, and 10).

Each level of this hierarchy contains interacting generalizations of (1.1a, b) as prototypes of the categories

(a) the *c-fixed deformations*

(b) the *c-changing deformations*

within the hierarchy, which would separately describe generalized twists and ghosts. We mention in particular the analysis of Section 10 which interprets the c-changing deformations as auxiliary compactified spacetime dimensions, effectively doubling the original compactified dimensions of the string.

2. Deformations

A continuous family of (chiral) conformal field theories is a representation of the Virasoro [5] algebra

$$(L_m[D], L_n[D]) = (m-n)\, L_{m+n}[D] + \frac{c[D]}{12} m(m^2-1)\delta_{m,-n} \tag{2.1}$$

whose elements are continuous functionals of a set of deformation parameters $\{D_{-p,A}\}$ where $p \in \mathbb{R}$ is a mode number and A is an internal index. An alternate viewpoint is that each representation of the algebra (2.1) defines a connected target-space $L_m[\mathbb{D}]$ of (chiral) conformal field theories on a connected, possibly infinite-dimensional, base-space of deformations $\mathbb{D}$ whose coordinates are $\{D_{-p,A}\}$.

Our strategy for the construction of such families focuses on the infinitesimal conformal deformations $\delta L_m[D] = \sum_p \delta D_{-p,A} J^A_{m,p}[D]$ in terms of the bilocal deformation currents

$$J^A_{m,p}[D] \equiv \frac{\partial L_m[D]}{\partial D_{-p,A}} \tag{2.2}$$

which are the tangent vectors of the target-space. We assume also a (chiral) conformal field theory $L_m[0]$, $c[0]$ at an origin $[0]$ of $\mathbb{D}$ such that δD is arbitrary at that point. Then the linear condition

$$\begin{aligned}(L_m[0], J^A_{n,p}[0]) &- (L_n[0], J^A_{m,p}[0]) \\ &= (m-n)\, J^A_{m+n,p}[0] + \frac{1}{12}\delta_{m,-n} m(m^2-1) \frac{\partial c[0]}{\partial D_{-p,A}}\end{aligned} \tag{2.3}$$

defines all possible deformation directions about that theory, and completion to finite deformation may be studied.

As a first step, we search through all local deformation directions $J^A_{m,p}[0] = J^A_{m+p}[0]$ which are derivatives $J^A_r[0] = r^n \mathscr{T}^A_r(h)$ of an $(h, 0)$ conformal tensor

$$(L_m[0], \mathscr{T}^A_r(h)) = (m(h-1) - r)\, \mathscr{T}^A_{m+r}(h) \tag{2.4}$$

with $r \in \mathbb{R}$ at the origin. The complete list of solutions in this class is

$$J^A_{m+p}[0] = \begin{cases} \mathcal{T}^A_{m+p}(h=1) & (2.5a) \\ m\mathcal{T}^A_m(h)\delta_{p,0} & (2.5b) \\ m^3\mathcal{T}^A_m(h=-1)\delta_{p,0} & (2.5c) \end{cases}$$

and $(\partial c/\partial D_{-p,A})[0]=0$, which shows an infinite number of deformation directions for each $(1, 0)$ operator at the origin in (2.5a), and a deformation direction for any integer-moded tensor in (2.5b). We have not systematized the general solution[1] to (2.3), and we will not study all the solutions in (2.5), but our choice of deformation currents at the origin will generate non-tensor and bilocal deformations away from the origin, and eventually an example (see Section 7.2) of a continuous system in which a deformation current is never a tensor.

The applications in this paper are primarily limited to simultaneous deformations of type (2.5a) and (2.5b) with the currents [1, 2] $\mathcal{T}^A_m(h=1)=T^A_m$, $m\in\mathbb{Z}$ of affine g

$$(L_m[0], T^A_n) = -nT^A_{m+n}, \qquad (T^A_m, T^B_n) = if^{AB}{}_C T^C_{m+n} + kg^{AB}m\delta_{m,-n}, \qquad (2.6)$$

where internal indices run over dim g, k is the central charge, and g_{AB} is the Killing metric of g. The Sugawara construction of $L_m[0]$ for arbitrary level of g is given in Section 5. Then the finite deformation

$$L_m[d, D_0] = L_m[0] + \sum_p (d_{-p,A} + mD_{0,A}\delta_{p,0})T^A_{m+p} + \varepsilon_m[d, D_0] \qquad (2.7a)$$

$$\varepsilon_m[d, D_0] = \frac{1}{2}k\left[\sum_p d_{-p,A}d^A_{m+p} + 2mD_{0,A}d^A_m - D_{0,A}D^A_0\delta_{m,0}\right] \qquad (2.7b)$$

$$\mathbb{D}:\quad 0 = d_{-m,A}D_{0,B}f^{AB}{}_C \qquad (2.7c)$$

$$c(D_0) = c(0) - 12kD_{0,A}D^A_0 \qquad (2.7d)$$

with $p\in\mathbb{Z}$ and $d^A_m \equiv g^{AB}d_{m,B}$, $D^A_0 \equiv g^{AB}D_{0,B}$ is completed on the constrained space of deformations (2.7c). We call this construction the class of *flat* deformations,[2] distinguishing also the *c-fixed* deformations $d_{m,A}$ and the *c-changing* deformations $D_{0,A}$, which would separately describe twists and generalized ghosts.

[1] Many other solutions to (2.3) exist, including $J_{m+p}=(L_m[0], A_p)$, $\forall A_p$. As an example, the deformation direction $J_m = mL_m[0]$ is obtained with the choice $A_p=\delta_{p,0}L_0[0]$, but the finite completion $L_m(d_0)=\exp(imd_0)L_m[0]$ is unitary-equivalent to $L_m[0]$.

[2] The flat-deformation parameters can be taken as arbitrary functions $d_{m,A}(\bar{T}_0)$, $D_{0,A}(\bar{T}_0)$ of any mutually commuting set $\{\bar{T}^a_0\}$ of constants of the motion, which commute with all the operators of the construction. The notation recalls that zero-modes of (0, 1) Cartan subalgebra currents are often available in a full conformal field theory.

The generally bilocal deformation currents $\partial L_m[d, D_0]/\partial d_{-p,A}$ and $\partial L_m[d, D_0]/\partial D_{0,A}$ in

$$\delta L_m[d, D_0] = \sum_p \delta d_{-p,A} \frac{\partial L_m[d, D_0]}{\partial d_{-p,A}} + \delta D_{0,A} \frac{\partial L_m[d, D_0]}{\partial D_{0,A}} \tag{2.8a}$$

$$(\delta d_{-p,A} D_{0,B} + d_{-p,A} \delta D_{0,B}) f^{AB}{}_C = 0 \tag{2.8b}$$

satisfy an analogous form of the linear condition (2.3) about the point $[d, D_0]$, but no path outside the space of deformations $\mathbb{D}$ in (2.7c) is obtained by following these directions, since the finite deformation

$$L_m[d+\hat{d}, D_0+\hat{D}_0] = L_m[d, D_0] + \sum_p \hat{d}_{-p,A} \frac{\partial L_m[d, D_0]}{\partial d_{-p,A}} + \hat{D}_{0,A} \frac{\partial L_m[d, D_0]}{\partial D_{0,A}} + \varepsilon_m[\hat{d}, \hat{D}_0] \tag{2.9a}$$

$$(\hat{d}_{-p,A} D_{0,B} + d_{-p,A} \hat{D}_{0,B} + \hat{d}_{-p,A} \hat{D}_{0,B}) f^{AB}{}_C = 0 \tag{2.9b}$$

is a quadratic form in the coordinates of $\mathbb{D}$. Among these deformation currents, only the currents $T[d]$ about $[d, 0]$

$$L_m[d+\hat{d}, D_0] = L_m[d, 0] + \sum_p (\hat{d}_{-p,A} + m D_{0,A} \delta_{p,0}) T^A_{m+p}[d] + \varepsilon_m[\hat{d}, D_0] \tag{2.10a}$$

$$T^A_{m+p}[d] \equiv \frac{\partial L_m[d, 0]}{\partial d_{-p,A}} = T^A_{m+p} + k d^A_{m+p} \tag{2.10b}$$

are local, and, discussed in Section 3, these are not generally tensors of the deformation away from the origin.

When the deformations are taken on the Cartan subalgebra (CSA) of g $(A \to a = 1, \ldots, \operatorname{rank} g)$ we may also allow the deformation parameters $d_{m,a}(T_0)$, $D_{0,a}(T_0)$ to be arbitrary functions of the zero-modes T^a_0 of the Cartan currents, a procedure which we call loading the deformation parameters. The resulting *arbitrarily loaded* deformations

$$L_m[d(T_0), D_0(T_0)] = L_m[0] + \sum_p (d_{-p,a}(T_0) + m D_{0,a}(T_0) \delta_{p,0}) T^a_{m+p} + \varepsilon_m[d(T_0), D_0(T_0)] \tag{2.11a}$$

$$c(D_0(T_0)) = c(0) - 12k D_{0,a}(T_0) D^a_0(T_0) \tag{2.11b}$$

satisfy the constraint (2.7c) and may be rotated off the CSA (see Appendix B). We remark here that these deformations generally involve operator-valued conformal weights (see Section 9) as well as the operator central charge (2.11b) (see Section 10), and that known mechanisms of spontaneous breakdown (see Section 8) are included in the c-fixed *linear-loaded* zero-mode deformations with only $d_0 = e + fT_0 \neq 0$.

3. Flat Deformations

The local formulation of the flat deformations in (2.7) is

$$L[d, D_0, \theta] = L[0, \theta] + (d_A(\theta) + D_{0,A}\, i\partial_\theta) T^A(\theta) + \varepsilon[d, D_0, \theta] \tag{3.1a}$$

$$\varepsilon[d, D_0, \theta] = \tfrac{1}{2}k(d_A(\theta) d^A(\theta) + 2D_{0,A}\, i\partial_\theta d^A(\theta) - D_{0,A} D_0^A) \tag{3.1b}$$

$$\mathbb{D}\colon\ D^A_{0,B}(T_{\text{adj}}) d^B(\theta) = d^A{}_B(T_{\text{adj}}, \theta) D_0^B = 0, \tag{3.1c}$$

where $\mathbb{O}(z) = \sum_m \mathbb{O}_m z^{-m}$, $z = \exp(i\theta)$ for any quantity $\mathbb{O}$, and

$$D^A_{0,B}(T_{\text{adj}}) \equiv (D_{0,C} T^C_{\text{adj}})^A{}_B, \qquad d^A{}_B(T_{\text{adj}}, \theta) \equiv (d_C(\theta) T^C_{\text{adj}})^A{}_B \tag{3.2}$$

with $(T^A_{\text{adj}})^B{}_C \equiv -if^{AB}{}_C$. In this section, we discuss explicitly only compact g, with $g^{AB} = \delta^{AB}$, $f^{AB}{}_C = f^{ABC}$, $d_A = d^A$, and $D_{0,A} = D_0^A$.

We first seek the tensors of the deformation. Natural candidates are the local deformation currents $T^A[d, z]$ about $[d, 0]$ in (2.10b), but these are not tensors, since

$$(L_m[d, D_0], T^A[d, z]) = z^m (D^{AB}[d, z, T_{\text{adj}}] + m h^{AB}(D_0, T_{\text{adj}})) \times T^B[d, z] + z^m k m^2 D_0^A \tag{3.3a}$$

$$D^{AB}[d, z, T_{\text{adj}}] \equiv \delta^{AB} z\partial_z - d^{AB}(T_{\text{adj}}, z), \qquad h^{AB}(D_0, T_{\text{adj}}) \equiv \delta^{AB} - D_0^{AB}(T_{\text{adj}}), \tag{3.3b}$$

where D^{AB} and h^{AB} are a g-covariant derivative and a conformal weight-matrix, respectively.

The covariant derivative in (3.3a) may be removed by introducing the orthogonal twist-matrix Ω,

$$\Omega[d, \theta, T_{\text{adj}}] \equiv \Theta^* \exp\left\{ -i \int_0^\theta d\phi\, d(T_{\text{adj}}, \phi) \right\}, \tag{3.4}$$

which is an anti-θ-ordered Wilson integral satisfying $i\partial_\theta \Omega = \Omega d$, and

$$\Omega^{AB}[d, \theta, T_{\text{adj}}] D_0^B = D_0^A \tag{3.5a}$$

$$(\Omega[d, \theta, T_{\text{adj}}], h(D_0, T_{\text{adj}})) = 0 \tag{3.5b}$$

according to the constraint (3.1c). The first relation and (3.3b) show that D_0 is a simultaneous eigenvector of Ω and h, while the second relation, that Ω commutes with h, is proven from the first in Appendix B. The twist-matrix is used to define boosted[3] current operators $T_{\text{boost}}[d] = \Omega T[d]$ or

$$T^A_{\text{boost}}[d, \theta] = \Omega^{AB}[d, \theta, T_{\text{adj}}](T^B(\theta) + kd^B(\theta)) \tag{3.6a}$$

$$= 2\pi \Omega^{AB}[d, \theta, T_{\text{adj}}] \frac{\delta}{\delta d^B(\theta)} L_0[d, D_0] \tag{3.6b}$$

which verify

$$(L_m[d, D_0], T^A_{\text{boost}}[d, z]) = z^m(\delta^{AB} z\partial_z + mh^{AB}(D_0, T_{\text{adj}})) \times T^B_{\text{boost}}[d, z] + z^m km^2 D_0^A \tag{3.7}$$

with (3.3a) and (3.5). The twist- and weight-matrices will determine the modeings and conformal-weights, respectively, of T_{boost}, but we call attention to the extra c-changing Schwinger term in (3.7) which cannot be further absorbed in any local T_{boost}.

According to (3.5b), we may introduce the $[d, D_0]$-dependent simultaneous eigenbasis $\{U_{\sigma/\mathbb{Z}, j}(A)\}$ of $\Omega(2\pi)$ and h,

$$\Omega^{AB}[d, 2\pi, T_{\text{adj}}] U_{\sigma/\mathbb{Z}, j}(B) = e^{2\pi i \sigma} U_{\sigma/\mathbb{Z}, j}(A) \tag{3.8a}$$

$$h^{AB}(D_0, T_{\text{adj}}) U_{\sigma/\mathbb{Z}, j}(B) = h_{\sigma/\mathbb{Z}, j}(D_0) U_{\sigma/\mathbb{Z}, j}(A) \tag{3.8b}$$

with $\sigma/\mathbb{Z} \equiv \sigma \bmod \mathbb{Z} = \sigma - \text{int}(\sigma)$ and $j = 1, ..., d(\sigma/\mathbb{Z})$, where $d(\sigma/\mathbb{Z})$ is the degeneracy of the eigenvalue $\exp(2\pi i\sigma)$. The basis is unitary

$$\sum_{A=1}^{\dim g} U^*_{\sigma/\mathbb{Z}, j}(A) U_{\sigma'/\mathbb{Z}, k}(A) = \delta_{(\sigma - \sigma')/\mathbb{Z}, 0} \delta^{jk} \tag{3.9a}$$

$$\sum_{0 \leqslant \sigma < 1} \sum_{j=1}^{d(\sigma/\mathbb{Z})} U^*_{\sigma/\mathbb{Z}, j}(A) U_{\sigma/\mathbb{Z}, j}(B) = \delta^{AB} \tag{3.9b}$$

and we take the convention $U_{\sigma=0, j=1}(A) = D_0^A/|D_0|$ which labels the simultaneous eigenvector D_0 as the first with $\sigma = 0$. It follows that the Cartesian frame is unitary-equivalent to a *completely homogeneous frame* in which the $[d, D_0]$-dependent twist-eigenstates

$$T^{\sigma/\mathbb{Z}, j}_{\text{boost}}[d, D_0, \theta] \equiv U_{\sigma/\mathbb{Z}, j}(A) T^A_{\text{boost}}[d, \theta] \tag{3.10a}$$

$$T^{\sigma/\mathbb{Z}, j}_{\text{boost}}[d, D_0, \theta + 2\pi] = e^{-2\pi i \sigma} T^{\sigma/\mathbb{Z}, j}_{\text{boost}}[d, D_0, \theta] \tag{3.10b}$$

have modeing $\sigma/\mathbb{Z}$, the weight-matrix is diagonal

$$(L_m[d, D_0], T^{\sigma/\mathbb{Z}, j}_{\text{boost}}[d, D_0, z]) = z^m(z\partial_z + mh_{\sigma/\mathbb{Z}, j}(-D_0)) T^{\sigma/\mathbb{Z}, j}_{\text{boost}}[d, D_0, z] + z^m km^2 |D_0| \delta_{\sigma/\mathbb{Z}, 0} \delta^{j,1}, \tag{3.11}$$

[3] $T^A_{\text{boost}}[d, \theta] = \exp(i\theta L_0[d, D_0]) T^A[d, 0] \exp(-i\theta L_0[d, D_0])$ is the SL_2-boost [2] of $T^A[d, 0]$.

and all operators except $T_{\rm boost}^{\sigma=0, j=1}$ are tensors. It also follows from (3.8) that

$$\forall i\, \exists j: \quad 2 = h_{\sigma/\mathbb{Z},i}(D_0) + h_{(-\sigma)/\mathbb{Z},j}(D_0) \tag{3.12}$$

so all conformal-weight shifts are paired as $h_{(\pm\sigma)/\mathbb{Z}} = 1 \pm \Delta_\sigma$.

The existence of a completely homogeneous frame for each flat deformation followed as a direct consequence of conformal invariance, which dictated the constrained space of deformations $\mathbb{D}$ in (2.7c).

The next question is the algebra of $\{T_{\rm boost}^A\}$. The continuous local automorphism[4]

$$(T_{\rm boost}^A[d,\theta], T_{\rm boost}^B[d,\theta']) = 2\pi i(f^{ABC}T_{\rm boost}^C[d,\theta] + \delta^{AB}k\partial_\theta)\delta(\theta-\theta') \tag{3.13a}$$

$$\delta(\theta-\theta') = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{im(\theta-\theta')} \tag{3.13b}$$

is verified on the fundamental range $|\theta|$, $|\theta'| < \pi$ with the conjugation identity of Appendix B. Then, the Fourier analysis of the local automorphism in the completely homogenous frame (3.10a)

$$\begin{aligned}(T_{\rm boost}^{\sigma/\mathbb{Z},j}[d,D_0]_{m+\sigma}, T_{\rm boost}^{\sigma'/\mathbb{Z},k}[d,D_0]_{n+\sigma'}) &= if_{(\sigma+\sigma')/\mathbb{Z},l}^{\sigma/\mathbb{Z},j;\sigma'/\mathbb{Z},k}\, T_{\rm boost}^{(\sigma+\sigma')/\mathbb{Z},l}[d,D_0]_{m+n+\sigma+\sigma'}\\ &\quad + k(m+\sigma)\,\delta_{m+\sigma,-n-\sigma'}\delta_{(\sigma+\sigma')/\mathbb{Z},0}\Delta_{\sigma/\mathbb{Z}}^{jk}\end{aligned} \tag{3.14a}$$

$$T_{\rm boost}^{\sigma/\mathbb{Z},j}[d,D_0]_{m+\sigma} \equiv \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\, e^{i(m+\sigma)\theta}T_{\rm boost}^{\sigma/\mathbb{Z},j}[d,D_0,\theta] \tag{3.14b}$$

$$f_{(\sigma+\sigma')/\mathbb{Z},l}^{\sigma/\mathbb{Z},j;\sigma'/\mathbb{Z},k} \equiv U_{\sigma/\mathbb{Z},j}(A)\,U_{\sigma'/\mathbb{Z},k}(B)\,f^{ABC}U^*_{(\sigma+\sigma')/\mathbb{Z},l}(C) \tag{3.14c}$$

$$\Delta_{\sigma/\mathbb{Z}}^{jk} \equiv U_{\sigma/\mathbb{Z},j}(A)\,U_{(-\sigma)/\mathbb{Z},k}(A) \tag{3.14d}$$

is a continuous global automorphism. The convention $U^*_{\sigma/\mathbb{Z},j} = U_{(-\sigma)/\mathbb{Z},j}$ may be adopted in the subspace with $h(D_0)=1$, so that $\Delta_{\sigma/\mathbb{Z}}^{jk} = \delta^{jk}$ in this case.

The modes of the invariant twist-class $\sigma/\mathbb{Z}=0$ [30] define the invariant affine subalgebra g_0 of the global automorphism (3.14), while boosted operators with non-zero $\sigma/\mathbb{Z}$ transform as representations of g_0 with $d(\sigma/\mathbb{Z})$-dimensional representation matrices

$$(T^I(\sigma/\mathbb{Z}))_k^j = -if_{\sigma/\mathbb{Z},k}^{0,I;\sigma/\mathbb{Z},j} \tag{3.15}$$

for $I = 1, ..., \dim g_0$.

It is particularly easy to extend the discussion to other tensor representations at the origin which do not have Schwinger terms with the deformation currents. In

[4] The existence of the local automorphism (3.13) was noted in [29] without an explicit form of $T_{\rm boost}$. The equivalent form $T_{\rm boost}^A(\theta)T_{\rm adj}^A = \Omega(\theta)(T^A(\theta)T_{\rm adj}^A - ik\partial_\theta)\Omega^\dagger(\theta)$ of (3.6) shows that the automorphism group of (3.13) is the local gauge-group $\Omega[d,\theta]\in G$, whose generators $\partial_0\mathcal{L}^A(\theta) \equiv 2\pi i\partial_\theta(\Omega^{AB}(\theta)\delta/\delta d^B(\theta))$ verify the classical current-algebra (3.13) with $k=0$.

this case, an $(h, 0)$ tensor R^i transforming in any Hermitian representation $(T^A)_{ij}$ of g satisfies

$$(L_m[0], R^i(z)) = z^m(z\partial_z + mh)R^i(z), \qquad R^i(z) = \sum_{p \in \mathbb{Z} + \sigma_0} R^i_p z^{-p}, \tag{3.16a}$$

$$(T^A_m, R^i(z)) = z^m R^j(z)(T^A)_{ji} = -z^m(\bar{T}^A)_{ij} R^j(z), \tag{3.16b}$$

where $\bar{T} = -T^*$. Then, the appropriate unitary left-twist-matrix for representation T commutes with the corresponding weight-matrix

$$\Omega[d, \theta, \bar{T}] \equiv \Theta^* \exp\left\{-i\int_0^\theta d\phi\, d(\bar{T}, \phi)\right\}, \qquad d^{ij}(\bar{T}, \theta) \equiv d^A(\theta)\, \bar{T}^A_{ij} \tag{3.17a}$$

$$(\Omega[d, \theta, \bar{T}], h(D_0, \bar{T})) = 0, \qquad h^{ij}(D_0, \bar{T}) \equiv h\delta^{ij} - D_0^A \bar{T}^A_{ij} \tag{3.17b}$$

and the conformal transformation

$$(L_m[d, D_0], R^i_{\text{boost}}[d, z]) = z^m(\delta^{ij} z\partial_z + mh^{ij}(D_0, \bar{T}))R^j_{\text{boost}}[d, z] \tag{3.18a}$$

$$R^i_{\text{boost}}[d, \theta] \equiv \Omega^{ij}[d, \theta, \bar{T}] R^j(\theta) \tag{3.18b}$$

is verified with no extra Schwinger term.

The relations (3.17) and (3.18a) guarantee a completely homogeneous frame in which the boosted operators are tensors. Introduce the $[d, D_0]$-dependent unitary eigenbasis $\{V_{\rho/\mathbb{Z},r}(i)\}$ with degeneracy label r,

$$\Omega^{ij}[d, 2\pi, \bar{T}]\, V_{\rho/\mathbb{Z},r}(j) = e^{2\pi i\rho} V_{\rho/\mathbb{Z},r}(i) \tag{3.19a}$$

$$h^{ij}(D_0, \bar{T})\, V_{\rho/\mathbb{Z},r}(j) = h_{\rho/\mathbb{Z},r}(D_0)\, V_{\rho/\mathbb{Z},r}(i), \tag{3.19b}$$

so that $\bar{V}_{(-\rho)/\mathbb{Z},r}(i) \equiv V^*_{\rho/\mathbb{Z},r}(i)$ is the right eigenbasis of $\Omega(2\pi, T)$ and $h(T)$. The result

$$(L_m[d, D_0], R^{\sigma/\mathbb{Z},r}_{\text{boost}}[d, D_0]_{n+\sigma}) = [(h_{\rho/\mathbb{Z},r}(D_0) - 1)m - (n+\sigma)]\, R^{\sigma/\mathbb{Z},r}_{\text{boost}}[d, D_0]_{m+n+\sigma} \tag{3.20a}$$

$$R^{\sigma/\mathbb{Z},r}_{\text{boost}}[d, D_0, \theta] \equiv \bar{V}_{(-\rho)/\mathbb{Z},r}(i) R^i_{\text{boost}}[d, \theta], \qquad \sigma = \sigma_0 - \rho \tag{3.20b}$$

follows with modes defined on the fundamental range.

As an example, consider antiperiodic complex world-sheet (Weyl) fermions [2, 12]

$$(\psi^i_p, \bar{\psi}^j_q)_+ = \delta^{ij}\delta_{p,-q}, \qquad p, q \in \mathbb{Z} + \tfrac{1}{2} \tag{3.21}$$

which satisfy (3.16) with $\bar{R}^i = \psi^i$, $R^i = \bar{\psi}^i$ in Hermitian representation $\bar{T}, T$. The boosted fermions

$$\psi^{\sigma/\mathbb{Z},r}_{\text{boost}}[d, D_0, \theta] = V_{\rho/\mathbb{Z},r}(i)\psi^i_{\text{boost}}[d, \theta], \qquad \bar{\psi}^{\sigma'/\mathbb{Z},r}_{\text{boost}}[d, D_0, \theta] = \bar{V}_{(-\rho)/\mathbb{Z},r}(i)\bar{\psi}^i_{\text{boost}}[d, \theta] \tag{3.22}$$

with $\sigma = \frac{1}{2} + \rho$, $\sigma' = \frac{1}{2} - \rho$ have definite conformal-weight and verify the continuous automorphism

$$(\psi^{\sigma/\mathbb{Z},r}_{\text{boost}}[d, D_0, \theta], \bar{\psi}^{\sigma'/\mathbb{Z},s}_{\text{boost}}[d, D_0, \theta'])_+ = 2\pi\delta_{(\sigma+\sigma')/\mathbb{Z},0}\delta^{r,s}\delta(\theta-\theta') \tag{3.23a}$$

$$(\psi^{\sigma/\mathbb{Z},r}_{\text{boost}}[d, D_0]_{m+\sigma}, \bar{\psi}^{\sigma'/\mathbb{Z},s}_{\text{boost}}[d, D_0]_{n+\sigma'})_+ = \delta_{(\sigma+\sigma')/\mathbb{Z},0}\delta^{r,s}\delta_{m+\sigma,-n-\sigma'} \tag{3.23b}$$

with θ, θ' and the modes defined on the fundamental range.

When R_i has Schwinger terms with the deformation currents (see Section 7), R^i_{boost} will both twist and shift in analogy with the currents (3.6a).

4. Twist-Pictures

4.1. *Torus-Picture and the Fixed-State Phenomenon*

The c-fixed CSA (torus) deformations $L_m[\tilde{d}]$ for arbitrary level of g at the origin are obtained by setting $\tilde{d}^A_m = (\tilde{d}^a_m, 0)$, $a=$ CSA, and $D_0 = 0$ in (2.7), but we focus on the zero-mode deformation

$$L_m(\tilde{d}_0) \equiv L_m[0] + \tilde{d}^a_0 T^a_m + \frac{1}{2} k\tilde{d}^2_0 \delta_{m,0} \tag{4.1.1a}$$

$$= e^{-\Lambda} L_m[\tilde{d}] e^{\Lambda}, \qquad \Lambda \equiv \sum_{m \neq 0} \frac{1}{m} \tilde{d}^a_{-m} T^a_m \tag{4.1.1b}$$

since the higher modes $\tilde{d}^a_{m\neq 0}$ are removable by unitary transformation. The twist-matrix $\Omega = \exp(-i\theta\tilde{d}_0 \cdot T_{\text{adj}})$ in this case gives the homogeneous $(1, 0)$ operators and their modes

$$T^a_{\text{boost}}(\tilde{d}_0, \theta) = T^a(\theta) + k\tilde{d}^a_0, \qquad E^\alpha_{\text{boost}}(\tilde{d}_0, \theta) \equiv \chi_\alpha(i) T^i_{\text{boost}}(\tilde{d}_0, \theta) = e^{i\theta\alpha \cdot \tilde{d}_0} E^\alpha(\theta) \tag{4.1.2a}$$

$$T^a_{\text{boost}}(\tilde{d}_0)_m = T^a_m + k\tilde{d}^a_0 \delta_{m,0}, \qquad E^\alpha_{\text{boost}}(\tilde{d}_0)_{m-\alpha\cdot\tilde{d}_0} = E^\alpha_m \tag{4.1.2b}$$

in the weight-basis $\{\chi_\alpha\}$ of the adjoint (see Appendix A), which is a special case of the basis $\{U_{\sigma/\mathbb{Z},j}\}$ with $\sigma_\alpha(\tilde{d}_0) = -\alpha \cdot \tilde{d}_0$.

An affine-conformal module [2, 30, 18] at the origin consists of a highest-weight state $|h, \mu\rangle$ (satisfying $L_{m>0} = T^A_{m>0} = E^{\alpha>0}_0 = 0$, $L_0 = h$, and $T^a_0 = \mu^a$ with μ a highest weight of g) and the states formed by multiple application of $E^{\alpha<0}_0$, $L_{m<0}[0]$, and $T^A_{m<0}$ on $|h, \mu\rangle$. A deformed module is defined in the same way by operation with $(E^{\alpha<0}_{\text{boost}})_0$ and the negative-modes of $T^A_{\text{boost}}(\vec{d}_0)$ and $L(\vec{d}_0)$ on a deformed highest-weight state which is an eigenstate of $L_0(\vec{d}_0)$ and $(T^a_{\text{boost}})_0$ with $L_{m>0}(\vec{d}_0) = (T^A_{\text{boost}})_{r>0} = (E^{\alpha>0}_{\text{boost}})_0 = 0$. The deformed module exhibits a *fixed-state phenomenon*: Since the states of the undeformed module have definite values of T^a_0, the same states remain eigenstates of $L_0(\vec{d}_0)$ throughout the deformation, and the deformed module is a continuous relabeling of the module at the origin.

The fixed-state phenomenon is observed for all the zero-mode deformations d^A_0, $D^A_0 \neq 0$ of the paper (see also Sections 6.3 and 10), since the deformation terms in $L_0(d_0, D_0)$ commute with $L_0[0]$.

The characterization of any particular fixed state as a highest-weight state may change at a point of degeneracy. For example, constructions with antiperiodic fermions [2] have the ground-state $|0\rangle_{\text{BH}} = |h, \mu = 0\rangle$ which is a highest-weight state with $E^\alpha_0 |0\rangle_{\text{BH}} = 0$ and $E^\alpha_{-1} |0\rangle_{\text{BH}} \neq 0$. This state remains a highest-weight state so long as $|\alpha \cdot \vec{d}_0| < 1$ for all α: When $\alpha \cdot \vec{d}_0 < -1$ for a particular α the positive-moded operator $E^\alpha_{\text{boost}}(\vec{d}_0)_{-1-\alpha \cdot \vec{d}_0} = E^\alpha_{-1}$ does not annihilate $|0\rangle_{\text{BH}}$. The highest-weight state at and beyond the degeneracy point is determined as follows: Although not highest weight for $|\alpha \cdot \vec{d}_0| < 1$, the states

$$(E^\alpha_{-1})^n |0\rangle_{\text{BH}} = (E^\alpha_{\text{boost}}(\vec{d}_0)_0)^n |0\rangle_{\text{BH}}, \qquad n \geqslant 1 \tag{4.1.3}$$

are degenerate with $|0\rangle_{\text{BH}}$ at $\alpha \cdot \vec{d}_0 = -1$. Any such family terminates after $n-1$ applications of the raising operator

$$0 = \|(E^\alpha_{-m})^n |h, \mu\rangle\|^2_0 = n\left(km - \alpha \cdot \mu - \frac{n-1}{2}\alpha^2\right) \|(E^\alpha_{-m})^{n-1} |h, \mu\rangle\|^2_0 \tag{4.1.4}$$

when the module has a positive norm at the origin.[5] In the present case $n = 1 + 2k/\alpha^2$, and it is verified that $|0\rangle_{BH}$ is the highest-weight state at $\alpha \cdot \vec{d}_0 = -1$ when $\alpha < 0$; the state

$$(E^\alpha_{-1})^{2k/\alpha^2} |0\rangle_{\text{BH}} = (E^\alpha_{\text{boost}}(\vec{d}_0)_{-1-\alpha \cdot \vec{d}_0})^{2k/\alpha^2} |0\rangle_{\text{BH}} \tag{4.1.5}$$

is the highest-weight state at $\alpha \cdot \vec{d}_0 = -1$ when $\alpha > 0$, and becomes the new highest-weight state for $\alpha \cdot \vec{d}_0 < -1$, until the twisted modeing of another boosted step-operator vanishes.

The algebra of the boosted currents (4.1.2b)

$$(T^a_{\text{boost}}(\vec{d}_0)_m, T^b_{\text{boost}}(\vec{d}_0)_n) = km\delta^{ab}\delta_{m,-n} \tag{4.1.6a}$$

$$(T^a_{\text{boost}}(\vec{d}_0)_m, E^\alpha_{\text{boost}}(\vec{d}_0)_{n-\alpha \cdot \vec{d}_0}) = \alpha^a E^\alpha_{\text{boost}}(\vec{d}_0)_{m+n-\alpha \cdot \vec{d}_0} \tag{4.1.6b}$$

$$(E^\alpha_{\text{boost}}(\vec{d}_0)_{m-\alpha \cdot \vec{d}_0}, E^\beta_{\text{boost}}(\vec{d}_0)_{n-\beta \cdot \vec{d}_0})$$
$$= \begin{cases} N(\alpha, \beta)\, E^{\alpha+\beta}_{\text{boost}}(\vec{d}_0)_{m+n-(\alpha+\beta) \cdot \vec{d}_0}, & \alpha + \beta = \text{root} \\ \alpha \cdot T_{\text{boost}}(\vec{d}_0)_{m+n} + k(m - \alpha \cdot \vec{d}_0)\delta_{m,-n}, & \alpha + \beta = 0 \\ 0, & \text{otherwise} \end{cases} \tag{4.1.6c}$$

[5] The positive number $n = 1 + 2\,\|E^\alpha_{-m} |h, \mu\rangle\|^2_0/\alpha^2 = 1 + 2(km - \alpha \cdot \mu)/\alpha^2$ is an integer since $2k/\alpha^2$ and $2\alpha \cdot \mu/\alpha^2$ are integers.

is a simple example of the continuous global automorphism (3.14). The form (4.1.6) and the connection (4.1.2b) were obtained along different lines for the order-N inner-automorphisms $N\alpha \cdot \tilde{a}_0 \in \mathbb{Z}$ ($\forall\alpha$) by Goddard and Olive [18]. The explicit form of the finite-order automorphisms is

$$\tilde{a}_0^\alpha = \frac{2}{N} \sum_{i=1}^{\text{rank } g} \frac{q_i \lambda_{(i)}^a}{\alpha_{(i)}^2}, \qquad q_i \in \mathbb{Z}, \tag{4.1.7}$$

where $\lambda_{(i)}$ and $\alpha_{(i)}$ are fundamental weights and simple roots, respectively, of g, and at least one q_i is relatively prime to N. The boosted step-operators E_{boost}^α with $\alpha = \sum_i n_i(\alpha)\alpha_{(i)}$, $n_i(\alpha) \in \mathbb{Z}$, then fall into twist-class [30] $p(\alpha) = (\sigma_\alpha(\tilde{a}_0)/\mathbb{Z})N = [-\sum_i q_i n_i(\alpha)] \bmod N$ between zero and $N-1$.

We have also examined the unitary-equivalent higher-mode deformation $L_m[\tilde{a}]$ in (4.1.1b), for which $T_{\text{boost}}^a[\tilde{a}, \theta] = T^a(\theta) + k\tilde{a}^a(\theta)$ and an extra factor $\exp[\sum_{m \neq 0} (1 - \exp(-im\theta))\alpha \cdot \tilde{a}_m/m]$ appears in (4.1.2a) for $E_{\text{boost}}^\alpha[\tilde{a}, \theta]$. The local automorphism (3.13) implies that the continuous global automorphism (4.1.6) is unmodified as expected, since the modeing of the operators does not change.

Similarly, the twist-matrix (3.17a) for the arbitrary representation (3.16) at the origin is $\exp(-i\theta\tilde{a}_0 \cdot \bar{T})$. The $(h, 0)$ twist-eigenstates are labelled by the weights of the representation

$$R_{\text{boost}}^\mu(\tilde{a}_0, \theta) \equiv \bar{\chi}_\mu(i) R_{\text{boost}}^i(\tilde{a}_0, \theta) = e^{i\theta\mu \cdot \tilde{a}_0} R^\mu(\theta) \tag{4.1.8}$$

in the weight-basis of Appendix A, which is a special case of the basis $\{\bar{V}_{(-\rho)/\mathbb{Z}, r}\}$ with $\sigma_\mu(\tilde{a}_0) = \sigma_0 - \mu \cdot \tilde{a}_0$.

As application, consider the currents $T^A(\theta) = \bar{\psi} T^A \psi$ [2] of the antiperiodic fermions (3.21) at the origin, and re-express the boosted currents (4.1.2b) in terms of the modes $\bar{\psi}_{\text{boost}}^\mu(\tilde{a}_0)_r$, $r \in \mathbb{Z} + \sigma_\mu(\tilde{a}_0)$, of the boosted fermions (4.1.8), which satisfy (3.23). The form of the boosted step-operators is unchanged since no shift and no re-normal-ordering with respect to the boosted modes is required. The result for the boosted Cartan currents

$$\begin{aligned} T_{\text{boost}}^a(\tilde{a}_0) &= \sum_\mu \mu^a :\bar{\psi}_{\text{boost}}^\mu(\tilde{a}_0) \psi_{\text{boost}}^{-\mu}(\tilde{a}_0):_{\text{re}} \\ &\quad + \sum_\mu \mu^a [\mu \cdot \tilde{a}_0 + \text{int}(-\mu \cdot \tilde{a}_0 + \tfrac{1}{2}) + \tfrac{1}{2}\delta_{\mu \cdot \tilde{a}_0, \mathbb{Z}+1/2}] \end{aligned} \tag{4.1.9}$$

is obtained with the relation $k\delta^{ab} = \text{Tr}\, T^a T^b = \Sigma_\mu \mu^a \mu^b$ in the form $k\tilde{a}_0^a = \Sigma_\mu \mu \cdot \tilde{a}_0 \mu^a$ and the re-normal-ordering prescription which antisymmetrizes zero-modes. The generally non-vanishing constant term in (4.1.9) is zero for integer and half-integer modeing of the fermions. Similarly the relation

$$L(\tilde{a}_0) = \frac{i}{2} \sum_{j=1}^{\dim T} :\bar{\psi}^j \bar{\partial}_\theta \psi^j: + \tilde{a}_0^a :\bar{\psi} T^a \psi: + \frac{1}{2} k\tilde{a}_0^2 \tag{4.1.10a}$$

$$= \frac{i}{2} \sum_\mu :\bar{\psi}_{\text{boost}}^\mu(\tilde{a}_0) \bar{\partial}_\theta \psi_{\text{boost}}^{-\mu}(\tilde{a}_0):_{\text{re}} + \frac{1}{2} \sum_\mu \left[\mu \cdot \tilde{a}_0 - \text{int}\left(\mu \cdot \tilde{a}_0 + \frac{1}{2}\right)\right] \tag{4.1.10b}$$

re-expresses the torus-deformation of the free-fermion construction [2] in terms of the boosted fermion modes. The absence of a linear term in the conventional form (4.1.10b) should be contrasted with the corresponding form of the twisted Sugawara constructions in Section 5.1.

The original $SU(3)$ deformation (1.1a) of Bardakci and Halpern [2] is a simple example of a conformal construction with a continuously twisted affine Lie algebra. The positive roots of $SU(3)$ are the two simple roots $\alpha_{\pm} = (1/\sqrt{2}, \mp\sqrt{3/2})$ and their sum β. The affine $SU(3)$ at the origin breaks to $g_0 = SU(2) \otimes U(1)$ since $E^{\pm\beta}_{\text{boost}}(\tilde{d}_0)_m = E^{\pm\beta}_m$, $T^3_{\text{boost}}(\tilde{d}_0)_m = T^3_m$, $T^8_{\text{boost}}(\tilde{d}_0)_m = T^8_m + \tilde{d}^8_0 \delta_{m,0}$ and the boosted step-operators of the simple roots $E^{\alpha\pm}_{\text{boost}}(\tilde{d}_0)_{m\pm\sigma}$, $\sigma = \tilde{d}^8_0 \sqrt{3/2}$, twist in opposite directions. Similarly, an application of (4.1.8) to the fermions in $\bar{3}$ gives $\psi^i_{\text{boost}}(\tilde{d}_0) = \exp[i\theta\rho_i(\tilde{d}_0)]\psi^i$ with $\rho_1 = \rho_2 = -\tilde{d}^8_0/\sqrt{6}$ and $\rho_3 = \tilde{d}^8_0\sqrt{2/3}$. Deformation of a free Bose–Fermi system (see also Section 6.3) is analyzed as another example in Appendix C.

4.2. *Orbifold-Picture*

An inner-automorphic orbifold-type twist [31, 32, 30, 33] is a view of a finite-order torus-twist (4.1.1a), (4.1.7) from a frame in which not all the CSA currents are integer-moded. These twists correspond to particular deformations of the form $\hat{d}^A_0 = (\hat{d}^a_0, \hat{d}^i_0 \neq 0)$ in our formulation: The rotation identity (D.5) of Appendix D in the form

$$T^A_{\text{boost}}(d_0, \theta) = \Gamma^{Ab}(T_{\text{adj}})\, T^{b,\text{torus}}_{\text{boost}}(\tilde{d}_0, \theta) + \sum_{\alpha} \Gamma^{Ai}(T_{\text{adj}})\, \chi_{-\alpha}(i)\, E^{\alpha,\text{torus}}_{\text{boost}}(\tilde{d}_0, \theta) \tag{4.2.1}$$

expresses the boosted currents of an arbitrary frame $d^A_0 = \Gamma^{Ab}(T_{\text{adj}})\,\tilde{d}^b_0$ as a rotation $\Gamma \in G$ of the torus-twisted currents with $\tilde{d}^A_0 = (\tilde{d}^a_0, 0)$. As an example (see also Appendix F), a (complete) orbifold-involution occurs when it is possible to choose $d_0 = \hat{d}_0$ so that all the Cartan currents on the left of (4.2.1) are half-integer moded,[6]

$$T^a_{\text{boost}}(\hat{d}_0, \theta) = \sum_{\substack{\alpha > 0 \\ p(\alpha) = 1}} \Gamma^{ai}(T_{\text{adj}})(\chi_{-\alpha}(i)\, E^{\alpha,\text{torus}}_{\text{boost}}(\tilde{d}_0, \theta) + \chi_{\alpha}(i)\, E^{-\alpha,\text{torus}}_{\text{boost}}(\tilde{d}_0, \theta)), \tag{4.2.2}$$

which requires $\Gamma^{ab} = 0$ and hence the off-Cartan form $\hat{d}^A_0 = (0, \hat{d}^i_0)$. The deformation also involves $\hat{d}^a_0 \neq 0$ when the involution is not complete, and homogeneous CSA components T^p in twist-class p of $T^a_{\text{boost}}(\hat{d}_0)$ are employed in general twisted vertex-operator constructions [32, 33].

We have also considered c-fixed deformation by the CSA components $T^{p\neq 0}$ of an inner- or outer-automorphism at the origin, but such deformations are unitary-removable as in (4.1.1b).

[6] Twisted-scalar fields were introduced in [34, 35].

4.3. *Magnetic-Analogue Picture*

The right-twist-matrix $\Omega[d, \theta, T]$ with $\bar{T} \to T$ in (3.17a) for general flat c-fixed deformations and Hermitian representation T of g solves the θ-dependent Schrödinger equation

$$i\partial_\theta \Omega^\dagger = H\Omega^\dagger, \qquad H = -d^A(\theta)T^A \tag{4.3.1}$$

which corresponds to the evolution of a non-Abelian "spin" T in an arbitrary magnetic field $B^A(\theta) = d^A(\theta)$ with period 2π. The zero-modes $d_0^A \to \bar{d}_0^a$ rotated onto the CSA are generalized Larmor frequencies of the system, while the higher modes provide rotating fields with integer angular frequencies. The evidence of Appendix E indicates that these deformations are equivalent to those of the other pictures.

5. Deformation of the Sugawara and Coset Constructions

5.1. *Twisted Sugawara Constructions*

We remark further on the continuous torus-twist of the Sugawara construction [8, 2, 9, 10, 18] for simple g,

$$L_m(\bar{d}_0) = \frac{1}{2k+Q_\psi} \sum_{n\in\mathbb{Z}} {}^\times_\times \left(\sum_{a=1}^{\text{rank}\, g} T^a_{m+n} T^a_{-n} + \sum_\alpha E^\alpha_{m+n} E^{-\alpha}_{-n} \right) {}^\times_\times + \bar{d}_0^a T^a_m + \frac{1}{2} k \bar{d}_0^2 \delta_{m,0} \tag{5.1.1a}$$

$${}^\times_\times A_m B_n {}^\times_\times \equiv \theta(n>0) A_m B_n + \theta(n \leqslant 0) B_n A_m, \qquad c(\bar{d}_0) = c(0) = \frac{2k \dim g}{2k+Q_\psi}, \tag{5.1.1b}$$

where Q_ψ is the quadratic Casimir in the adjoint. A more conventional form of the construction is obtained by re-expressing $L_m(\bar{d}_0)$ in terms of the boosted modes with the inverse of (4.1.2b). The result is

$$\begin{aligned} L_m(\bar{d}_0) = {} & \frac{1}{2k+Q_\psi} \sum_{n\in\mathbb{Z}} \sum_{a=1}^{\text{rank}\, g} {}^\times_\times T^a_{\text{boost}}(\bar{d}_0)_{m+n} T^a_{\text{boost}}(\bar{d}_0)_{-n} {}^\times_\times \\ & + \frac{1}{2k+Q_\psi} \sum_{n\in\mathbb{Z}} \sum_\alpha {}^\times_\times E^\alpha_{\text{boost}}(\bar{d}_0)_{m+n-\alpha\cdot\bar{d}_0} E^{-\alpha}_{\text{boost}}(\bar{d}_0)_{-n+\alpha\cdot\bar{d}_0} {}^\times_{\times\,\text{re}} \\ & + \frac{1}{2k+Q_\psi} \left[Q_\psi \bar{d}_0^a - \sum_\alpha \alpha^a \operatorname{int}(\alpha\cdot\bar{d}_0) \right] T^a_{\text{boost}}(\bar{d}_0)_m + \varepsilon(\bar{d}_0)_{\text{re}} \delta_{m,0} \end{aligned} \tag{5.1.2a}$$

$$\begin{aligned} {}^\times_\times E^\alpha_{\text{boost}}(\bar{d}_0)_{m-\alpha\cdot\bar{d}_0} E^{-\alpha}_{\text{boost}}(\bar{d}_0)_{n+\alpha\cdot\bar{d}_0} {}^\times_{\times\,\text{re}} & \\ \equiv \theta(n+\alpha\cdot\bar{d}_0 > 0) & E^\alpha_{\text{boost}}(\bar{d}_0)_{m-\alpha\cdot\bar{d}_0} E^{-\alpha}_{\text{boost}}(\bar{d}_0)_{n+\alpha\cdot\bar{d}_0} \\ + \theta(n+\alpha\cdot\bar{d}_0 \leqslant 0) & E^{-\alpha}_{\text{boost}}(\bar{d}_0)_{n+\alpha\cdot\bar{d}_0} E^\alpha_{\text{boost}}(\bar{d}_0)_{m-\alpha\cdot\bar{d}_0} \end{aligned} \tag{5.1.2b}$$

$$\varepsilon(\bar{d}_0)_{\text{re}} = \frac{k}{2k+Q_\psi} \left(\sum_\alpha \alpha\cdot\bar{d}_0 \operatorname{int}(\alpha\cdot\bar{d}_0) - \frac{1}{2} Q_\psi \bar{d}_0^2 - \frac{1}{2} \sum_\alpha \operatorname{int}(\alpha\cdot\bar{d}_0)(\operatorname{int}(\alpha\cdot\bar{d}_0)+1) \right) \tag{5.1.2c}$$

after using the boosted current algebra (4.1.6) to re-normal-order with respect to the boosted modes. The generally non-vanishing term linear in $T^a_{\text{boost}}(\tilde{a}_0)$ collects contributions from three sources: the original linear term, the $k\tilde{a}^a_0$ shift terms of the inversion (4.1.2b), and the re-normal-ordering.

The condition for the vanishing of the linear term in (5.1.2a)

$$\tilde{a}^a_0 = \frac{1}{Q_\psi} \sum_\alpha \alpha^a \operatorname{int}(\alpha \cdot \tilde{a}_0) \tag{5.1.3}$$

is interesting since it selects a certain class of finite-order twists for which the construction simplifies, as analyzed below.

A weak upper bound $N \leqslant 2Q_\psi/\psi_s^2$ on the allowed order N of such twists, with ψ_s a short root of g, is obtained by comparison of (5.1.3) and (4.1.7), and solutions $\tilde{a}_0$ of (5.1.3) fall into representations of the Weyl group of g because $W^a_\alpha(\tilde{a}_0) \equiv \tilde{a}^a_0 - 2\alpha^a \alpha \cdot \tilde{a}_0/\alpha^2$ is also a solution. Moreover, the ρ-solution

$$\tilde{a}_0(\rho) = \frac{2}{Q_\psi} \rho, \qquad \rho = \sum_{i=1}^{\operatorname{rank} g} \lambda_{(i)} = \frac{1}{2} \sum_{\alpha > 0} \alpha, \qquad |\alpha \cdot \tilde{a}_0(\rho)| < 1 \tag{5.1.4}$$

and its Weyl-transforms are always available, since $\operatorname{sign} \alpha \cdot \tilde{a}_0 = \operatorname{sign} \alpha$ in this case and $Q_\psi = (\psi + 2\rho) \cdot \psi$ with ψ the highest root of g.

Completeness of the root system $\sum_\alpha \alpha^a \alpha^b = Q_\psi \delta^{ab}$ in the form $Q_\psi \tilde{a}^a_0 = \sum_\alpha \alpha \cdot \tilde{a}_0 \alpha^a$ recasts the condition (5.13) as

$$\sum_{\substack{\alpha > 0 \\ p(\alpha) > 0}} \left(1 - \frac{2p(\alpha)}{N}\right) \alpha = 0, \tag{5.1.5}$$

where $p(\alpha) = (\sigma_\alpha(\tilde{a}_0)/\mathbb{Z}) N = -N[\alpha \cdot \tilde{a}_0 + \operatorname{int}(-\alpha \cdot \tilde{a}_0)]$ is the twist-class of the step-operator E^α_{boost}. It follows that the twisted Sugawara construction (5.1.2) is free of linear terms for any involution.

Another form of (5.1.3) is

$$0 = \sum_{\substack{p=1 \\ p \neq N/2}}^{N-1} \left(1 - \frac{p}{N}\right) \sum_{i=1}^{d(p/N)} f_{0,J}^{p/N,i;1-p/N,i} = \sum_{\substack{p=1 \\ p \neq N/2}}^{N-1} \left(1 - \frac{p}{N}\right) \operatorname{Tr}\left(T^J\left(\frac{p}{N}\right)\right) \tag{5.1.6}$$

for all $J \in g_0$, in terms of the twisted structure constants (3.14c) and traces of the representation matrices (3.15). Absence of the linear term follows when $T^J(p/N)$ is traceless for all p, which is satisfied when g_0 is semi-simple.

Kac and Peterson [36] have given a Sugawara construction for all finite-order automorphisms. We have checked that the continuously twisted Sugawara construction (5.1.1) or (5.1.2) agrees with their construction in the region of overlap, namely for finite-order inner-automorphisms. In particular, the linear term obtained by normal-ordering their construction is proportional to the form (5.1.6)—so no linear term will be required for outer-automorphisms [37], since g_0 is simple in these cases.

5.2. *Twisted Coset Constructions*

The G/H coset construction [2, 9, 34, 11, 18] with $c_K = c_g - c_h$ is

$$K_m[0] = L^g_m[0] - L^h_m[0], \qquad (K_m[0], T^a_n) = 0, \tag{5.2.1}$$

where $L^{g,h}[0]$ are the Sugawara constructions of $g \supset h$ at the origin and T^a_m are the currents of h. When g and h are simple the remaining currents T^I_m in g/h satisfy

$$(K_m[0], T^I_n) = -nK^{IJ}T^J_{m+n} - \frac{if^{aIJ}}{2k + Q^h_\psi} \sum_{l \in Z} {}^\times_\times (T^a_{m+n+l} T^J_{-l} + T^J_{m+n+l} T^a_{-l}) {}^\times_\times \tag{5.2.2a}$$

$$(K_m[0], T^I_n) - (K_n[0], T^I_m) = (m-n)K^{IJ}T^J_{m+n} \tag{5.2.2b}$$

with

$$K^{IJ} = \delta^{IJ} - \frac{Q^{IJ}}{2k + Q^h_\psi}, \qquad Q^{IJ} = \sum_{a=1}^{\dim h} \sum_{L \in g/h} f^{IaL} f^{JaL}. \tag{5.2.3}$$

Comparing (5.2.2b) with the linear condition (2.3) gives the allowed c-fixed deformations[7] of the coset construction

$$K_m[d] = K_m[0] + \sum_{(r), I = g/h} d^{(r)}_{-n} \mathscr{L}^{(r)}_I T^I_{m+n} + \tfrac{1}{2}k \sum_{(r)} d^{(r)}_{-n} d^{(r)}_{m+n}, \tag{5.2.4}$$

where $\{\mathscr{L}^{(r)}_I\}$ are the orthonormal null-eigenvectors of Q^{IJ}. The form $b_I Q^{IJ} b_J$ is non-negative for all b_I, so the deformation currents $\{\mathscr{L}^{(r)}_I T^I\}$ commute with the currents of h and transform as $(1, 0)$ tensors under L^g and K. Equivalently, the set of deformation currents is the centralizer of h in g/h. The same steps are followed and the same conclusion is obtained for arbitrary G/H.

A similar conclusion is reached for noncompact coset constructions [9] beginning with the Sugawara construction [2, 9, 38]

$$L^g[0, z] = \frac{1}{2k + Q^g_\psi} \sum_{A=1}^{\dim g} {}^\times_\times T^A(z) T_A(z) {}^\times_\times, \qquad Q^g_\psi \delta^A{}_B = f^{AC}{}_D f_{BC}{}^D \tag{5.2.5}$$

for non-compact simple g with $c(0) = 2k \dim g/(2k + Q^g_\psi)$. In this case, the deformation currents $\mathscr{L}^{(r)}_I T^I$ with $\mathscr{L}_I Q^I{}_J = 0$ include the centralizer of h in g/h ((1, 0) tensors of K), but may extend beyond the centralizer since $b_I Q^I{}_J b^J$ can be negative.

[7] c-changing deformation of a coset construction is not possible with the original currents of the construction, since the quadratic terms of (5.2.2a) for nT^I_n cannot satisfy the linear condition (2.3).

As an illustration, we discuss the original[8] coset constructions [9]

$$SO(4,2)\supset SO(4,1)\supset SO(3,1) \tag{5.2.6}$$

in which the (spinor representation) level-one $SO(4,2)$ currents were expressed as the (four-dimensional) spacetime-fermion [2, 21] bilinears $T(\Gamma)=\bar{\psi}\Gamma\psi=(T,A,V,P)$ and their real fermion (defining representation) equivalents $T(\Gamma)\sim(b_\mu,b_\nu)$. The constructions (5.2.6) correspond to $TAVP\supset TA\supset T$. The (5, 6) rotation $P=\bar{\psi}i\gamma_5\psi$ commutes with $L^h=T^{\mu\nu}T_{\mu\nu}/48$ for $h=SO(3,1)$, so an allowed deformation of K for $SO(4,2)/SO(3,1)$ is

$$K_m[d]=\left(-\frac{1}{120}T^2+\frac{1}{40}(V^2-A^2-P^2)\right)_m+\sum_{n\in\mathbb{Z}}d_{m-n}P_n-2\sum_{n\in\mathbb{Z}}d_{m-n}d_n \tag{5.2.7}$$

in the original notation. The search for further deformation currents is completed with ordinary Diracology: $Q^I{}_J$ is block-diagonal with two 4×4 blocks (VV and AA) and a zero corresponding to P. The VV block is $Q^\rho_\lambda\sim\operatorname{Tr}\{\gamma^\rho(\sigma_{\mu\nu},(\sigma^{\mu\nu},\gamma_\lambda))\}\sim\delta^\rho{}_\lambda$ and similarly for the AA block, so there are no deformation currents outside the centralizer in this case. On the other hand, no further search is necessary in any compact construction, so the example $SO(m+n)/SO(m)$, $n\geqslant 2$, is deformable only with the currents of $SO(n)$.

We also remark that no deformations by the original currents of the construction are possible when g is compact and simple and G/H is a symmetric space, since the centralizer is an empty set. The same conclusion is verified for semi-simple g in the case $H\otimes H/\operatorname{diag}H$, which includes the $SU(2)$ construction of the (m,n) discrete series [40].

It follows more generally that there is no local (1, 0) current in any construction for which analysis of the Kac-determinant [41] shows discrete highest weights, since c-fixed deformation by that current would generate a continuous highest-weight spectrum without affecting the norm (see also Section 10).

6. Twists and Ghosts

6.1. *Torus-Ghosts*

We consider the general flat zero-mode deformation $L_m(d_0,D_0)$ after rotation of the c-fixed deformation onto the CSA. The resulting torus-ghosts

$$L_m(\tilde{d}_0,D_0)=L_m[0]+(\tilde{d}_0^A+mD_0^A)T_m^A+\tfrac{1}{2}k(\tilde{d}_0^2-D_0^2)\,\delta_{m,0} \tag{6.1.1a}$$

$$\tilde{d}_0^A=(\tilde{d}_0^a,0),\qquad D_0^A=\left(\tilde{D}_0^a,\sum_{\alpha\cdot\tilde{d}_0=0}D_\alpha\chi_\alpha(i)\right),\qquad c(D_0)=c(0)-12kD_0^2 \tag{6.1.1b}$$

[8] The $L_2=L-L_1$, $(L_1,L_2)=0$ phenomenon was discovered in the "spin-orbit" construction of [2], and coset constructions for discrete symmetry-breaking of $g=SU(3)$ were noted implicitly in [2, Sect. 3]. The first explicit coset constructions were the "spin–spin" interactions in Eq. (3.9) ($SO(4,2)\supset SO(4,1)$) and Eq. (3.14) ($SO(4,1)\supset SO(3,1)$) of [9]. Reference [34] suggests generalization of an example ($SO(4,1)\supset SO(3,1)$) of coset factorization $V_g=V_h\otimes V_{G/H}$ of tensors, which implies that K-degeneracy is also interpretable as multiplicative renormalization. More general discussion of these ideas is found in [39].

for arbitrary level of g at the origin are obtained by solving the constraint (2.7c) in this picture.[9] Here $\{\chi_\alpha\}$ is the weight-basis of the adjoint, $\{\tilde{d}_0^a\}$ is an arbitrary torus-twist, and $\{\tilde{D}_0^a\}$ is the generic c-changing deformation. The exceptional c-changing deformations $D_\alpha^* = D_{-\alpha}$ are interesting but difficult, since their completely homogeneous frame is not the Cartan–Weyl frame.

The discussion of Section 4.1 is extended with minor modifications for the generic c-changing deformations $\tilde{D}_0^A = (\tilde{D}_0^a, 0)$. The boosted currents (4.1.2) and their algebra (4.1.6) are unmodified, while the relations

$$(L_m(\tilde{d}_0, \tilde{D}_0), T^a_{\text{boost}}(\tilde{d}_0)_n) = -nT^a_{\text{boost}}(\tilde{d}_0)_{m+n} + km^2 \tilde{D}_0^a \delta_{m,-n} \tag{6.1.2a}$$

$$(L_m(\tilde{d}_0, \tilde{D}_0), E^\alpha_{\text{boost}}(\tilde{d}_0)_{n-\alpha\cdot\tilde{d}_0}) = (\alpha\cdot\tilde{D}_0 m - (n - \alpha\cdot\tilde{d}_0)) E^\alpha_{\text{boost}}(\tilde{d}_0)_{m+n-\alpha\cdot\tilde{d}_0} \tag{6.1.2b}$$

show conformal-weights $1+\alpha\cdot\tilde{D}_0$ for the boosted step-operators—and that one of the CSA currents is not a tensor, in accord with (3.11). Similarly, conformal-weight

$$h(\mu, \tilde{D}_0) = h + \mu\cdot\tilde{D}_0 \tag{6.1.3}$$

is found for the boosted form (4.1.8) of the arbitrary representation (3.16) at the origin. A proviso is that degeneracies such as (4.1.3) may be infinite when the norm at the origin is not positive, as seen in the explicit bosonic ghost constructions of Section 6.3. See also the further remark on norms in Section 10.

6.2. *Orbifold-Ghosts*

Any orbifold-twist $\hat{d}_0$ can similarly be promoted to an orbifold-ghost via the c-changing deformations $\hat{D}_0$ which satisfy $f^{ABC}\hat{d}_0^B \hat{D}_0^C = 0$, and an example is given in Appendix *F*. We remark in general that the CSA basis of an orbifold-ghost has indefinite conformal-weight since the conformal-weights of T^p and T^{N-p} shift in opposite directions according to (3.12).

6.3. $|SL_2\rangle_0$*-Preserving Deformations*

Any highest-weight state at the origin which is also an SL_2-invariant state

$$L_m[0]\,|SL_2\rangle_0 = 0, \qquad |m| \leqslant 1 \tag{6.3.1}$$

is automatically a singlet

$$T_0^A\,|SL_2\rangle_0 = (T_1^A, L_{-1}[0])\,|SL_2\rangle_0 = 0. \tag{6.3.2}$$

Define an $|SL_2\rangle_0$-preserving deformation as one which maintains the SL_2-character of the fixed state $|SL_2\rangle_0$ throughout the deformation. In the case of the generic

[9] The form $D_0^A = (D_0^a, 0)$, $d_m^A = (d_m^a, \sum_{\alpha\cdot D_0 = 0} d_m^\alpha \chi_\alpha(i))$ is the general solution of the constraint (2.7c) in the picture where the c-changing deformation is on the CSA.

c-changing deformation $L_m(\tilde{d}_0, \tilde{D}_0)$ in (6.1.1), the deformation with $\tilde{d}_0^a = \tilde{D}_0^a$ is $|SL_2\rangle_0$-preserving since[10]

$$L_m(\tilde{D}_0, \tilde{D}_0) = L_m[0] + (1+m)\tilde{D}_0 \cdot T_m, \qquad c(\tilde{D}_0) = c(0) - 12k\tilde{D}_0^2 \tag{6.3.3a}$$

$$L_m(\tilde{D}_0, \tilde{D}_0)\,|SL_2\rangle_0 = 0, \qquad |m| \leqslant 1 \tag{6.3.3b}$$

and the central charge may be considered as a function of the twist. Further classes of $|SL_2\rangle_0$-preserving deformations are noted below, and a unified mechanism is given in Section 10.

The ground-state of any free or Sugawara construction at the origin in terms of antiperiodic representations is always an $|SL_2\rangle_0$. We focus in this subsection on constructions with complex antiperiodic Fermi or Bose quarks [2]

$$\psi^i(z) = \sum_{r \in \mathbb{Z}+1/2} \psi_r^i z^{-r}, \qquad \bar{\psi}^i(z) = \sum_{r \in \mathbb{Z}+1/2} \bar{\psi}_r^i z^{-r} \tag{6.3.4a}$$

$$\psi_r^i \bar{\psi}_s^j + \tau \bar{\psi}_s^j \psi_r^i = \delta^{ij}\delta_{r,-s}, \qquad \tau = \begin{bmatrix} +1, & \text{Fermi} \\ -1, & \text{Bose} \end{bmatrix} \tag{6.3.4b}$$

$$\psi_r^i\,|0\rangle_{\mathrm{BH}} = \bar{\psi}_r^i\,|0\rangle_{\mathrm{BH}} = 0, \qquad r > 0 \tag{6.3.4c}$$

whose currents for Hermitian representation T [2],

$$T^A(z) = :\bar{\psi}^i(z)\, T_{ij}^A \psi^j(z):, \qquad k = \frac{\tau}{\dim g}\operatorname{Tr}(T^A T^A), \tag{6.3.5}$$

are reducible in the Fermi case when T is antisymmetric.[11]

It follows that the constructions [2, 42, 28, 43, 38]

$$L_{\text{free}}[0] = \frac{i}{2}\sum_{j=1}^{\dim T} :\bar{\psi}^j \bar{\partial}_\theta \psi^j: \rightarrow c_{\text{free}}(\tilde{D}_0) = \tau \dim T - 12k\tilde{D}_0^2 \tag{6.3.6a}$$

$$L_{\text{Sug}}[0] = \frac{1}{2k+Q_\psi}\sum_{A=1}^{\dim g} \overset{\times}{\underset{\times}{}} T^A T^A \overset{\times}{\underset{\times}{}} \rightarrow c_{\text{Sug}}(\tilde{D}_0) = \frac{2k\dim g}{2k+Q_\psi} - 12k\tilde{D}_0^2 \tag{6.3.6b}$$

$$L_{\text{torus}}[0] = \frac{1}{2k}\sum_{a=1}^{\operatorname{rank} g} \overset{\times}{\underset{\times}{}} T^a T^a \overset{\times}{\underset{\times}{}} \rightarrow c_{\text{torus}}(\tilde{D}_0) = \operatorname{rank} g - 12k\tilde{D}_0^2 \tag{6.3.6c}$$

in $L(\tilde{D}_0, \tilde{D}_0)$ of (6.3.3a) are $|SL_2\rangle_0$-preserving with $|SL_2\rangle_0 = |0\rangle_{\mathrm{BH}}$ fixed throughout the deformation. The deformations also share the boosted currents and current algebra (4.1.2) and (4.1.6) with $\sigma_\alpha(\tilde{D}_0) = 1 - h(\alpha, \tilde{D}_0) = -\alpha \cdot \tilde{D}_0$, and all

[10] Similarly, $L_m(d_0^A = \tilde{D}_0^A)$ is the general flat zero-mode $|SL_2\rangle_0$-preserving deformation, which includes $|SL_2\rangle_0$-preserving orbifold-ghosts and reproduces (6.3.3) after rotation onto the CSA.

[11] Bose–Fermi generalizations of the re-normal-ordered boosted currents (4.1.9) and free constructions (4.1.10b) are obtained with an extra factor τ multiplying the constant terms in these results, while the discussion of the twisted Sugawara and coset constructions in Section 5 is unchanged for negative k.

deficits[12] $c'(\tilde{D}_0)-c(\tilde{D}_0)$ are independent of $\tilde{D}_0$. The SL_2-state is no longer a highest-weight state when $|\alpha\cdot\tilde{D}_0|$ exceeds one for any root α (see Section 4.1): It is further verified for each Bose-module that the $(E^{\alpha}_{-m})^n$ degeneracy at $\alpha\cdot\tilde{D}_0=-m$ is infinite, since the indefinite norm at the origin does not terminate the family, and the spectrum is generically bottomless beyond the first degeneracy point. Moreover, the choice with $L_{\text{free}}[0]$ at the origin,

$$L(\tilde{D}_0,\tilde{D}_0)=L_{\text{free}}[0]+(1+i\partial_\theta)\tilde{D}_0\cdot T \tag{6.3.7a}$$

$$\psi^{\mu}_{\text{boost}}(\tilde{D}_0,\theta)=\chi_\mu(j)\psi^{j}_{\text{boost}}(\tilde{D}_0,\theta)=e^{i\theta\mu\cdot\tilde{D}_0}\chi_\mu(j)\psi^{j}(\theta) \tag{6.3.7b}$$

$$\bar{\psi}^{\mu}_{\text{boost}}(\tilde{D}_0,\theta)=\bar{\chi}_\mu(j)\bar{\psi}^{j}_{\text{boost}}(\tilde{D}_0,\theta)=e^{i\theta\mu\cdot\tilde{D}_0}\bar{\chi}_\mu(j)\bar{\psi}^{j}(\theta) \tag{6.3.7c}$$

$$h(\mu,\tilde{D}_0)=\tfrac{1}{2}+\mu\cdot\tilde{D}_0,\qquad \sigma_\mu(\tilde{D}_0)=\tfrac{1}{2}-\mu\cdot\tilde{D}_0, \tag{6.3.7d}$$

is the simplest non-Abelian generalization of the usual ghosts [28].

The case $g=U(1)$ in (6.3.7) has deficit $c_{\text{Sug}}(\tilde{D}_0)-c_{\text{free}}(\tilde{D}_0)=1-\tau$, and the SL_2-state is the ground-state of the deformation only for $|\tilde{D}_0|<\frac{1}{2}$ (see Appendix C). Those sectors of the usual ghosts [28] with $|SL_2\rangle_0=|0\rangle_{\text{BH}}$ are located at the points[13]

$$c=\psi^{\mu=-1}_{\text{boost}}(\tilde{D}_0=\tfrac{3}{2}),\ b=\bar{\psi}^{\mu=1}_{\text{boost}}(\tilde{D}_0=\tfrac{3}{2}),\ \tau=1 \qquad\text{(reparam)}$$

$$h(\bar{\psi})=h(b)=2,\ \sigma/\mathbb{Z}=0,\ c(\tilde{D}_0=\tfrac{3}{2})=-26 \tag{6.3.8}$$

and

$$\gamma=\psi^{\mu=-1}_{\text{boost}}(\tilde{D}_0=1),\ \beta=\bar{\psi}^{\mu=1}_{\text{boost}}(\tilde{D}_0=1),\ \tau=-1 \qquad\text{(NS)}$$

$$h(\bar{\psi})=h(\beta)=\tfrac{3}{2},\ \sigma/\mathbb{Z}=\tfrac{1}{2},\ c(\tilde{D}_0=1)=11. \tag{6.3.9}$$

The complex Ramond (CR) sector of the superconformal ghosts is located at $\tau=-1$, $\tilde{D}_0=1$ in the $\tilde{d}_0=\tilde{D}_0-\frac{1}{2}$ deformation

$$\begin{aligned}L^{\text{BH}}_m[0]+(\tilde{D}_0-\tfrac{1}{2})T_m+m\tilde{D}_0T_m+\tfrac{1}{2}(\tfrac{1}{4}-\tilde{D}_0)\delta_{m,0}\\ =L^{\text{CR}}_m[0]+(1+m)\tilde{D}_0(T_m-\tfrac{1}{2}\delta_{m,0})\end{aligned} \tag{6.3.10}$$

so that $h(\beta)=3/2$, $\sigma/\mathbb{Z}=0$, $c(\tilde{D}_0=1)=11$, and there is no SL_2-state. World-sheet supersymmetry of generalized ghost systems is discussed in the following section.

We finally remark on the vertex-operator constructions [12–14] of the currents

[12] The deficit $c_{\text{Sug}}(0)-c_{\text{free}}(0)$ was studied in [2] (level-one of $U(3)$ with T in the fundamental, and equivalence with free fermions), [9] (level-one of $SO(3,1)$ with T in the (Dirac) spinor representation = level-one of $SO(3,1)$ with T in the defining representation for real fermions, and equivalence with free fermions), and more recently in [44, 28, 28].

[13] The Hermiticity convention [28] $\psi^{\dagger}_{\text{boost}}(\tilde{D}_0)_{m+1/2+\tilde{D}_0}=\psi_{\text{boost}}(\tilde{D}_0)_{-(m+1/2+\tilde{D}_0)}$, $\bar{\psi}^{\dagger}_{\text{boost}}(\tilde{D}_0)_{m+1/2-\tilde{D}_0}=\tau\bar{\psi}_{\text{boost}}(\tilde{D}_0)_{-(m+1/2-\tilde{D}_0)}$ is an option when $\tilde{D}_0$ is integer or half-integer. Then $T^{\dagger}_m=-T_{-m}$, which is included in the usual affine algebra for Abelian g, and $L^{\dagger}_m(\tilde{D}_0,\tilde{D}_0)=L_{-m}(\tilde{D}_0,\tilde{D}_0)$ in this case.

(6.3.5) in the Fermi and Bose cases, following the original method of [12, 45]. The $i=1, ..., \dim T$ Fermi [12, 13, 15] and Bose [28] quarks are

$$(\text{Fermi}) \quad \psi^i = \xi_i z^{1/2} :e^{-iQ_S^i}:, \qquad \bar{\psi}^i = \xi_i z^{1/2} :e^{iQ_S^i}:, \qquad \xi_i = \prod_{j=1}^{i-1} (-1)^{\mathrm{int}\, T_{S,0}^j} \tag{6.3.11a}$$

$$(\text{Bose}) \quad \psi^i = :e^{-i(Q_T^i + Q_S^i)}:, \qquad \bar{\psi}^i = :(\partial_\theta Q_S^i + \tfrac{1}{2})\, e^{i(Q_T^i + Q_S^i)}:, \tag{6.3.11b}$$

where Q_T and Q_S are time-like and space-like, respectively, ξ_i is the Klein transformation (cocycle) [46, 12–14], and the normal-ordering notation [18] includes q-factors to the left of T_0-factors, which differs from the older notation [47]. The vertex-operator construction of the currents (6.3.5) is then

$$(\text{Fermi}) \quad T^A = \sum_i T_{ii}^A \partial_\theta Q_S^i + \sum_{i \neq j} T_{ij}^A c_{ij} z :e^{i\alpha_{ij} \cdot Q_S}: \tag{6.3.12a}$$

$$(\text{Bose}) \quad T^A = -\sum_i T_{ii}^A \partial_\theta Q_T^i + \sum_{i \neq j} T_{ij}^A :(\partial_\theta Q_S^i + \tfrac{1}{2}) e^{i\alpha_{ij} \cdot (Q_T + Q_S)}:, \tag{6.3.12b}$$

where $\alpha_{ij} = e_i - e_j$ are the roots of $SU(\dim T)$ and $c_{ij} = \xi_i \xi_j \exp[i\pi\theta(i>j)]$. These constructions form Bose–Fermi pairs with level $\pm 2\,|k|/\psi^2$ for arbitrary representation T of g. The following cases of (6.3.12) are known: level 1 of $SU(N)$ [12, 13] with T in the fundamental, (two copies of) level 1 of $SO(2N)$ [15] when T is in the defining representation and level -1 of $SU(2)$ [43] with T in the fundamental. Moreover, the Fermi construction (6.3.12a) is presumably related to the constructions in [48].

7. Flat Superconformal Deformations

7.1. $N=1$ (*Non-linear*) SUSY

Begin at the origin with the non-linear $N=1$ supersymmetric system [49–52]

$$L[0] = \frac{i}{2} {}^{\circ}_{\circ} S^A \bar{\partial}_\theta S^A {}^{\circ}_{\circ} + \varepsilon[0], \qquad \varepsilon[0] = \begin{bmatrix} 0 & \mathrm{AP} \\ \frac{1}{16} \dim g & P \end{bmatrix} \tag{7.1.1a}$$

$$G[0] = \frac{i}{6\sqrt{k}} f^{ABC} {}^{\circ}_{\circ} S^A S^B S^C {}^{\circ}_{\circ} = -\frac{1}{3\sqrt{k}} {}^{\circ}_{\circ} T^A S^A {}^{\circ}_{\circ} \tag{7.1.1b}$$

$$T^A = \tfrac{1}{2} {}^{\circ}_{\circ} S^B (T_{\mathrm{adj}}^A)_{BC} S^C {}^{\circ}_{\circ}, \qquad c(0) = \tfrac{1}{2} \dim g, \qquad k = \tfrac{1}{2} Q_\psi \tag{7.1.1c}$$

for semi-simple g with structure constants f^{ABC} and periodic or antiperiodic fermions $S^A = \sum_{r \in \mathbb{Z} + \sigma_0} S_r^A z^{-r}$. The algebra at the origin consists of an $N=1$ superconformal system

$$(L_m[0], L_n[0]) = (m-n)\, L_{m+n}[0] + \frac{c(0)}{12} m(m^2-1)\delta_{m,-n} \tag{7.1.2a}$$

$$(L_m[0], G_r[0]) = \left(\frac{1}{2}m - r\right) G_{m+r}[0] \tag{7.1.2b}$$

$$(G_r[0], G_s[0])_+ = 2L_{r+s}[0] + \frac{c(0)}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s} \tag{7.1.2c}$$

and an auxiliary algebra [50, 51]

$$(L_m[0], T_n^A) = -nT_{m+n}^A, \qquad (L_m[0], S_r^A) = -(\tfrac{1}{2}m + r)S_{m+r}^A \tag{7.1.3a}$$

$$(G_r[0], T_m^A) = \sqrt{k}\, mS_{m+r}^A, \qquad (G_r[0], S_s^A)_+ = -\frac{1}{\sqrt{k}} T_{r+s}^A \tag{7.1.3b}$$

$$((T_m^A, S_r^B) = if^{ABC}S_{m+r}^C, \qquad (S_r^A, S_s^B)_+ = \delta^{AB}\delta_{r,-s} \tag{7.1.3c}$$

$$(T_m^A, T_n^B) = if^{ABC}T_{m+n}^C + km\delta^{AB}\delta_{m,-n} \tag{7.1.3d}$$

which restricts the representations, as in (7.1.1). The auxiliary algebra also provides all the currents in this case, since the $N=1$ algebra specifies none of its own.

Maintenance of supersymmetry throughout a deformation is generally a restriction on allowed deformations of the conformal subalgebra. Our approach to this problem begins with a general (trial) conformal deformation $L[d, D_0]$ which determines a tensor (trial) G_{boost} along the lines which led to (3.18b) for R_{boost} and (3.6a) for T_{boost}. In particular, the G-multiplet for an $N \geqslant 1$ SUSY algebra in representation T under the deformation currents will twist with the appropriate $\Omega[d, \theta, \bar{T}]$, and a shift, as in T_{boost}, will be necessary to cancel the contribution of any Schwinger term of the multiplet with the deformation currents. The requirement that G_{boost} has conformal-weight $\frac{3}{2}$ eliminates c-changing deformation by currents which rotate $G[0]$, since these automatically change the conformal-weight. If the resulting (trial) G_{boost} fails to generate the rest of the automorphism, the deformation must be further restricted until it does.

In the present case, $G[0]$ is a singlet under g, so G_{boost} will not twist or change its conformal-weight under deformation by the currents of g. On the other hand, a shift is necessary because (7.1.3b) involves a Schwinger term of the deformation currents with $G[0]$, whose intuitive origin is the factor T^A in [7.1.1b). The result

$$L_m[d, D_0] = L_m[0] + \sum_{n \in \mathbb{Z}} d_{-n}^A T_{m+n}^A + mD_0^A T_m^A + \varepsilon_m[d, D_0] \tag{7.1.4a}$$

$$G_{\text{boost}}[d, D_0] = \frac{i}{6\sqrt{k}} f^{ABC}\, S^A S^B S^C - \sqrt{k}\,(d^A S^A + 2iD_0^A \partial_\theta S^A) \tag{7.1.4b}$$

$$\varepsilon_m[d, D_0] = \frac{1}{2}k\left(\sum_{n \in \mathbb{Z}} d_{-n}^A d_{m+n}^A + 2md_m^A D_0^A - D_0^A D_0^A \delta_{m,0}\right) \tag{7.1.4c}$$

$$c(D_0) = c(0) - 12kD_0^A D_0^A \tag{7.1.4d}$$

verifies a c-changing $N=1$ superconformal automorphism without further restriction on the space of deformations $\mathbb{D}$ in (2.7c).

The fermions S^A twist (but do not shift) with the same $\Omega[d, \theta, T_{\mathrm{adj}}]$ as their superpartners T^A, so homogeneous operators for the entire system are easily found. A number of distortions in the auxiliary algebra are caused by $D_0^A \neq 0$, however, in analogy with (3.11).

We illustrate in the case of zero-mode CSA deformations $\tilde{d}_0^a$, $\tilde{D}_0^a \neq 0$, for which the boosted operators are the currents in (4.1.2) and

$$S^a_{\mathrm{boost}}(\tilde{d}_0, \theta) \equiv S^a(\theta) \tag{7.1.5a}$$

$$S^\alpha_{\mathrm{boost}}(\tilde{d}_0, \theta) \equiv \chi_\alpha(i) S^i_{\mathrm{boost}}(\tilde{d}_0, \theta) = e^{i\theta\alpha \cdot \tilde{d}_0} S^\alpha(\theta) \tag{7.1.5b}$$

with the weight-basis $\{\chi_\alpha\}$ of Appendix A. Then the boosted auxiliary algebra

$$(L_m(\tilde{d}_0, \tilde{D}_0), S^a_{\mathrm{boost}}(\tilde{d}_0)_r) = -(\tfrac{1}{2}m + r)\, S^a_{\mathrm{boost}}(\tilde{d}_0)_{m+r} \tag{7.1.6a}$$

$$(*)\ (L_m(\tilde{d}_0, \tilde{D}_0), S^\alpha_{\mathrm{boost}}(\tilde{d}_0)_{r-\alpha\cdot\tilde{d}_0}) = [(\alpha \cdot \tilde{D}_0 - \tfrac{1}{2})m - (r - \alpha \cdot \tilde{d}_0)] \times S^\alpha_{\mathrm{boost}}(\tilde{d}_0)_{m+r-\alpha\cdot\tilde{d}_0} \tag{7.1.6b}$$

$$(G_{\mathrm{boost}}(\tilde{d}_0, \tilde{D}_0)_r, T^a_{\mathrm{boost}}(\tilde{d}_0)_m) = \sqrt{k}\, m S^a_{\mathrm{boost}}(\tilde{d}_0)_{m+r} \tag{7.1.6c}$$

$$(*)\ (G_{\mathrm{boost}}(\tilde{d}_0, \tilde{D}_0)_r, E^\alpha_{\mathrm{boost}}(\tilde{d}_0)_{m-\alpha\cdot\tilde{d}_0}) = \sqrt{k}\, [m - \alpha \cdot \tilde{d}_0 - 2r\alpha \cdot \tilde{D}_0] \times S^\alpha_{\mathrm{boost}}(\tilde{d}_0)_{m+r-\alpha\cdot\tilde{d}_0} \tag{7.1.6d}$$

$$(*)\ (G_{\mathrm{boost}}(\tilde{d}_0, \tilde{D}_0)_r, S^a_{\mathrm{boost}}(\tilde{d}_0)_s)_+ = -\frac{1}{\sqrt{k}} T^a_{\mathrm{boost}}(\tilde{d}_0)_{r+s} - 2\sqrt{k}\, r\tilde{D}_0^a \delta_{r,-s} \tag{7.1.6e}$$

$$(G_{\mathrm{boost}}(\tilde{d}_0, \tilde{D}_0)_r, S^\alpha_{\mathrm{boost}}(\tilde{d}_0)_{s-\alpha\cdot\tilde{d}_0})_+ = -\frac{1}{\sqrt{k}} E^\alpha_{\mathrm{boost}}(\tilde{d}_0)_{r+s-\alpha\cdot\tilde{d}_0} \tag{7.1.6f}$$

$$\begin{aligned}(S^a_{\mathrm{boost}}(\tilde{d}_0)_r, E^\alpha_{\mathrm{boost}}(\tilde{d}_0)_{m-\alpha\cdot\tilde{d}_0}) &= (T^a_{\mathrm{boost}}(\tilde{d}_0)_m, S^\alpha_{\mathrm{boost}}(\tilde{d}_0)_{r-\alpha\cdot\tilde{d}_0}) \\ &= \alpha^a S^\alpha_{\mathrm{boost}}(\tilde{d}_0)_{m+r-\alpha\cdot\tilde{d}_0}\end{aligned} \tag{7.1.6g}$$

$$(E^\alpha_{\mathrm{boost}}(\tilde{d}_0)_{m-\alpha\cdot\tilde{d}_0}, S^\beta_{\mathrm{boost}}(\tilde{d}_0)_{r-\beta\cdot\tilde{d}_0}) = \begin{cases} N(\alpha, \beta) S^{\alpha+\beta}_{\mathrm{boost}}(\tilde{d}_0)_{m+r-(\alpha+\beta)\cdot\tilde{d}_0}, & \alpha+\beta = \text{root} \\ \alpha \cdot S_{\mathrm{boost}}(\tilde{d}_0)_{m+r}, & \alpha+\beta = 0 \\ 0, & \text{otherwise} \end{cases} \tag{7.1.6h}$$

$$(S^a_{\mathrm{boost}}(\tilde{d}_0)_r, S^b_{\mathrm{boost}}(\tilde{d}_0)_s)_+ = \delta^{ab}\delta_{r,-s} \tag{7.1.6i}$$

$$(T^a_{\mathrm{boost}}(\tilde{d}_0)_m, S^b_{\mathrm{boost}}(\tilde{d}_0)_r) = (S^a_{\mathrm{boost}}(\tilde{d}_0)_r, S^\alpha_{\mathrm{boost}}(\tilde{d}_0)_{s-\alpha\cdot\tilde{d}_0})_+ = 0 \tag{7.1.6j}$$

$$(S^\alpha_{\mathrm{boost}}(\tilde{d}_0)_{r-\alpha\cdot\tilde{d}_0}, S^\beta_{\mathrm{boost}}(\tilde{d}_0)_{s-\beta\cdot\tilde{d}_0})_+ = \delta_{\alpha+\beta,0}\delta_{r,-s} \tag{7.1.6k}$$

is obtained, together with the $\{L, T\}$ and $\{T, T\}$ boosted subalgebras in (6.1.2) and

(4.1.6). Those relations which distort the auxiliary algebra (7.1.3) are denoted with $(*)$, but they are harmless since the basic $N=1$ SUSY is always preserved.

We also remark that a large number,

$$\dim SO(\dim g) - \dim g = \tfrac{1}{2}\dim g(\dim g - 3), \tag{7.1.7}$$

of "hidden" currents (including $\mathrm{int}(\frac{1}{2}\dim g) - \mathrm{rank}\, g$ hidden CSA currents), which generally fail to commute with $G[0]$ in (7.1.b), may be constructed as bilinears in S^A. These are the currents of hidden higher-N SUSYs within the $N=1$ model: The original $G[0]$ is a Cartesian component of the $N \geqslant 2$ G-multiplet and members of the multiplet are the twist-eigenstates of the hidden currents. Put another way, deforming the $N=1$ model with respect to such hidden currents will twist $G[0]$ to a G-multiplet, yielding deformed higher-N models as in the following section.

7.2. $N=2$ SUSY

We begin with an $N=2$ SUSY [7] at the origin,

$$(L_m[0], L_n[0] = (m-n)L_{m+n}[0] + \frac{c(0)}{12} m(m^2-1)\delta_{m,-n} \tag{7.2.1a}$$

$$(L_m[0], T_n) = -nT_{m+n}, \qquad (L_m[0], G_r^{\pm}[0]) = (\tfrac{1}{2}m - r)G_r^{\pm}[0] \tag{7.2.1b}$$

$$(T_m, T_n) = k(0)m\delta_{m,-n}, \qquad (T_m, G_r^{\pm}[0]) = \pm G_{m+r}^{\pm}[0] \tag{7.2.1c}$$

$$(G_r^{+}[0], G_s^{-}[0])_+ = 2L_{r+s}[0] + (r-s)T_{r+s} + \frac{c(0)}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s} \tag{7.2.1d}$$

$$(G_r^{+}[0], G_s^{+}[0])_+ = (G_r^{-}[0], G_s^{-}[0])_+ = 0, \qquad c(0) = 3k(0), \tag{7.2.1e}$$

where $m, n \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \frac{1}{2}$, leaving the twisted-sector (outer-automorphism) for separate discussion.

We also introduce an abstract auxiliary algebra at the origin, representations of which are discussed below:

$$\begin{aligned}(T_m^{\mathrm{aux},a}, \bar{T}_n^{\mathrm{aux},b}) &= m\hat{k}\delta^{ab}\delta_{m,-n},\\ (\psi_r^{\mathrm{aux},a}, \bar{\psi}_s^{\mathrm{aux},b})_+ &= \hat{k}\delta^{ab}\delta_{r,-s}\end{aligned} \tag{7.2.2a}$$

$$\begin{aligned}(L_m[0], T_n^{\mathrm{aux},a}) &= -nT_{m+n}^{\mathrm{aux},a} + \tfrac{1}{2}\lambda^a m^2\delta_{m,-n},\\ (L_m[0], \bar{T}_n^{\mathrm{aux},a}) &= -n\bar{T}_{m+n}^{\mathrm{aux},a} + \tfrac{1}{2}\bar{\lambda}^a m^2\delta_{m,-n}\end{aligned} \tag{7.2.2b}$$

$$\begin{aligned}(T_m, T_n^{\mathrm{aux},a}) &= \lambda^a m\delta_{m,-n},\\ (T_m, \bar{T}_n^{\mathrm{aux},a}) &= -\bar{\lambda}^a m\delta_{m,-n}\end{aligned} \tag{7.2.2c}$$

$$\begin{aligned}(G_r^{-}[0], T_m^{\mathrm{aux},a}) &= -\sqrt{2}\, m\psi_{r+m}^{\mathrm{aux},a},\\ (G_r^{+}[0], \bar{T}_m^{\mathrm{aux},a}) &= -\sqrt{2}\, m\bar{\psi}_{r+m}^{\mathrm{aux},a}\end{aligned} \tag{7.2.2d}$$

$$(G_r^+[0], \psi_s^{\text{aux},a})_+ = \sqrt{2}\,(T_{r+s}^{\text{aux},a} + \lambda^a r\delta_{r,-s}),$$
$$(G_r^-[0], \bar{\psi}_s^{\text{aux},a})_+ = \sqrt{2}\,(\bar{T}_{r+s}^{\text{aux},a} + \bar{\lambda}^a r\delta_{r,-s}) \tag{7.2.2e}$$

$$(G_r^+[0], T_m^{\text{aux},a}) = (G_r^-[0], \psi_s^{\text{aux},a})_+ = (G_r^-[0], \bar{T}_m^{\text{aux},a}) = (G_r^+[0], \bar{\psi}_s^{\text{aux},a})_+ = 0. \tag{7.2.2f}$$

The auxiliary fermions ψ^{aux}, $\bar{\psi}^{\text{aux}}$ at the origin are weight $\frac{1}{2}$, transform like G^-, G^+ under T, and commute with the auxiliary currents T^{aux}, $\bar{T}^{\text{aux}}$. Moreover, the auxiliary currents are not tensors when $\lambda^a, \bar{\lambda}^a \neq 0$, and the parameter $\hat{k}$ can be scaled to zero or one.

Following the method of Section 7.1, we begin with the general zero-mode[14] (trial) conformal deformation

$$L_m(D) = L_m[0] + (d_0 + mD_0)T_m + (\bar{d}_0^{\text{aux},a} + m\bar{D}_0^{\text{aux},a})T_m^{\text{aux},a}$$
$$+ \bar{T}_m^{\text{aux},a}(d_0^{\text{aux},a} + mD_0^{\text{aux},a}) + \varepsilon(D)\delta_{m,0} \tag{7.2.3a}$$

$$\varepsilon(D) = \tfrac{1}{2}k(0)d_0^2 + d_0(\bar{d}_0^{\text{aux},a}\lambda^a - \bar{\lambda}^a d_0^{\text{aux},a}) + \hat{k}(\bar{d}_0^{\text{aux},a} d_0^{\text{aux},a} - \bar{D}_0^{\text{aux},a} D_0^{\text{aux},a})$$
$$- \tfrac{1}{2}(\bar{D}_0^{\text{aux},a}\lambda^a + \bar{\lambda}^a D_0^{\text{aux},a}) - D_0(\bar{D}_0^{\text{aux},a}\lambda^a - \bar{\lambda}^a D_0^{\text{aux},a}) - \tfrac{1}{2}k(0)D_0^2 \tag{7.2.3b}$$

$$c(D) = c(0) - 12(\bar{D}_0^{\text{aux},a}\lambda^a + \bar{\lambda}^a D_0^{\text{aux},a}) - 24\hat{k}\bar{D}_0^{\text{aux},a} D_0^{\text{aux},a}$$
$$- 24D_0(\bar{D}_0^{\text{aux},a}\lambda^a - \bar{\lambda}^a D_0^{\text{aux},a}) - 12k(0)D_0^2 \tag{7.2.3c}$$

by the $N=2$ current T and the auxiliary currents. The distorted weights $h_\pm = \frac{3}{2} \pm D_0$ of the resulting (trial) supercharges

$$G_{\text{boost}}^+(D)_{r-d_0} = G_r^+[0] + \sqrt{2}\,\bar{\psi}_r^{\text{aux},a}\left(d_0^{\text{aux},a} + 2(r-d_0)\frac{D_0^{\text{aux},a}}{1+2D_0}\right) \tag{7.2.4a}$$

$$G_{\text{boost}}^-(D)_{r+d_0} = G_r^-[0] + \sqrt{2}\left(\bar{d}_0^{\text{aux},a} + 2(r+d_0)\frac{\bar{D}_0^{\text{aux},a}}{1-2D_0}\right)\psi_r^{\text{aux},a} \tag{7.2.4b}$$

reflect that $G^+[0]$, $G^-[0]$ are singlets under the auxiliary currents but rotate under T. The result at $D_0 = 0$,

$$L_m(D) = L_m[0] + d_0 T_m + (\bar{d}_0^{\text{aux},a} + m\bar{D}_0^{\text{aux},a})T_m^{\text{aux},a}$$
$$+ \bar{T}_m^{\text{aux},a}(d_0^{\text{aux},a} + mD_0^{\text{aux},a}) + \varepsilon(D)\delta_{m,0} \tag{7.2.5a}$$

$$T_{\text{boost}}(D)_m = T_m + 2\bar{T}_m^{\text{aux},a} D_0^{\text{aux},a} - 2\bar{D}_0^{\text{aux},a} T_m^{\text{aux},a} + \eta(D)\delta_{m,0} \tag{7.2.5b}$$

$$G_{\text{boost}}^+(D)_{r-d_0} = G_r^+[0] + \sqrt{2}\,\bar{\psi}_r^{\text{aux},a}(d_0^{\text{aux},a} + 2(r-d_0)D_0^{\text{aux},a}) \tag{7.2.5c}$$

$$G_{\text{boost}}^-(D)_{r+d_0} = G_r^-[0] + \sqrt{2}\,(\bar{d}_0^{\text{aux},a} + 2(r+d_0)\bar{D}_0^{\text{aux},a})\psi_r^{\text{aux},a} \tag{7.2.5d}$$

[14] Higher-mode deformations are removable with $\hat{A} = \sum_{m \neq 0}(d_{-m}T_m + \bar{d}_{-m}^{\text{aux},a}T_m^{\text{aux},a} + \bar{T}_m^{\text{aux},a}d_{-m}^{\text{aux},a})/m$.

$$\varepsilon(D)=\tfrac{1}{2}k(0)d_0^2+d_0(\bar{d}_0^{\text{aux},a}\lambda^a-\bar{\lambda}^a d_0^{\text{aux},a})$$
$$+\hat{k}(\bar{d}_0^{\text{aux},a}d_0^{\text{aux},a}-\bar{D}_0^{\text{aux},a}D_0^{\text{aux},a})-\tfrac{1}{2}(\bar{D}_0^{\text{aux},a}\lambda^a+\bar{\lambda}^a D_0^{\text{aux},a}) \tag{7.2.5e}$$

$$\eta(D)=k(0)d_0-2d_0(\bar{\lambda}^a D_0^{\text{aux},a}-\bar{D}_0^{\text{aux},a}\lambda^a)+\bar{d}_0^{\text{aux},a}\lambda^a-\bar{\lambda}^a d_0^{\text{aux},a}$$
$$+2\hat{k}(\bar{d}_0^{\text{aux},a}D_0^{\text{aux},a}-\bar{D}_0^{\text{aux},a}d_0^{\text{aux},a}) \tag{7.2.5f}$$

$$c(D)=c(0)-12(\bar{D}_0^{\text{aux},a}\lambda^a-\bar{\lambda}^a D_0^{\text{aux},a})-24\hat{k}\bar{D}_0^{\text{aux},a}D_0^{\text{aux},a}=3k(D), \tag{7.2.5g}$$

verifies a c-changing $N=2$ automorphism without further restriction on the deformation. The c-fixed representation-independent twist by T with only $d_0\neq 0$ was first given by Schwimmer and Seiberg [29].

The boosted auxiliary quantities

$$\psi_{\text{boost}}^{\text{aux},a}(D)_{r+d_0}=\psi_r^{\text{aux},a}, \qquad T_{\text{boost}}^{\text{aux},a}(D)_m=T_m^{\text{aux},a}+(\hat{k}d_0^{\text{aux},a}+\lambda^a d_0)\delta_{m,0} \tag{7.2.6a}$$

$$\bar{\psi}_{\text{boost}}^{\text{aux},a}(D)_{r-d_0}=\bar{\psi}_r^{\text{aux},a}, \qquad \bar{T}_{\text{boost}}^{\text{aux},a}(D)_m=\bar{T}_m^{\text{aux},a}+(\hat{k}\bar{d}_0^{\text{aux},a}-\bar{\lambda}^a d_0)\delta_{m,0} \tag{7.2.6b}$$

generate a number of harmless distortions in the boosted auxiliary algebra. In particular, the boosted auxiliary currents $T_{\text{boost}}^{\text{aux},a}(\bar{T}_{\text{boost}}^{\text{aux},a})$ are tensors of $L_m(D)$ only at the points $\lambda^a=-2\hat{k}D_0^{\text{aux},a}$ $(\bar{\lambda}^a=-2\hat{k}\bar{D}_0^{\text{aux},a})$, while the boosted auxiliary fermions remain tensors with $h=\frac{1}{2}$, since they commute with the auxiliary currents.

Similarly, c-fixed superconformal deformations by half-integer-moded currents T, T^{aux}, $\bar{T}^{\text{aux}}$ [34, 35] can be introduced in the twisted-sector with G_2 antiperiodic and G_1 periodic, but these deformations are unitary removable as in Section 4.

The simplest representation of the full $N=2$ algebra (7.2.1), (7.2.2) is the original construction [7]

$$L[0]=\sum_{b=1}^{M}\left(\,{}^{\times}_{\times}\,\bar{T}^{\text{aux},b}T^{\text{aux},b}\,{}^{\times}_{\times}+\frac{i}{2}\,{}^{\circ}_{\circ}\,\bar{\psi}^{\text{aux},b}\overleftrightarrow{\partial}_\theta\psi^{\text{aux},b}\,{}^{\circ}_{\circ}\right) \tag{7.2.7a}$$

$$G^+[0]=\sqrt{2}\sum_{b=1}^{M}\bar{\psi}^{\text{aux},b}T^{\text{aux},b}, \qquad G^-[0]=\sqrt{2}\sum_{b=1}^{M}\bar{T}^{\text{aux},b}\psi^{\text{aux},b} \tag{7.2.7b}$$

$$T=\sum_{b=1}^{M}\,{}^{\circ}_{\circ}\,\bar{\psi}^{\text{aux},b}\psi^{\text{aux},b}\,{}^{\circ}_{\circ} \tag{7.2.7c}$$

$$k(0)=M=\frac{c(0)}{3}, \qquad \hat{k}=1, \quad \lambda^b=\bar{\lambda}^b=0 \tag{7.2.7d}$$

in terms of the auxiliary fermions and currents. This representation is the hidden $N=2$ SUSY of the $N=1$ system (7.1.1) with $g=SU(2)^{2M}$ and $G^{N=1}[0]=(G^+[0]+G^-[0])/\sqrt{2}$. In particular, the auxiliary constructs are

$$\psi^{\text{aux},b}=-\frac{1}{\sqrt{2}}(S^{2b-1}+iS^{2b}), \qquad T^{\text{aux},b}=\frac{1}{2}(T^{2b-1}+iT^{2b}), \qquad b=1,\ldots,M \tag{7.2.8}$$

in terms of the Cartan fermions S^a and CSA currents T^a of that model, and the $N=2$ current is a sum over all the hidden CSA currents. The central charge

$$c(D)=3k(D)=3(M-8\bar{D}_0^{\mathrm{aux},b}D_0^{\mathrm{aux},b}) \tag{7.2.9}$$

is obtained for the corresponding deformation (7.2.5), (7.2.7).

Moreover, if we consider the point $D_0^{\mathrm{aux}}, \bar{D}_0^{\mathrm{aux}} \neq 0$, $d_0=d_0^{\mathrm{aux}}=\bar{d}_0^{\mathrm{aux}}=0$ in the representation (7.2.5), (7.2.7) above as a new origin in the space of deformations, then $T^{\mathrm{aux}}, \psi^{\mathrm{aux}}$ form a $\hat{k}=1$ representation of the auxiliary algebra (7.2.2) with $\lambda^a=2D_0^{\mathrm{aux},a}$, $\bar{\lambda}^a=2\bar{D}_0^{\mathrm{aux},a}$. Further deformations by the non-tensor currents T^{aux}, $\bar{T}^{\mathrm{aux}}$ about this origin remain in the original space (7.2.5), (7.2.7) of tensor deformations, as seen above in (2.9).

Another representation is a continuous generalization of the known $N=2$ superconformal ghost system [28]. Since our representation at the origin is not familiar, we give the final result

$$L(D)=\frac{i}{2}:\bar{\psi}^{\mathrm{F}}\overleftrightarrow{\partial}_\theta\psi^{\mathrm{F}}:+\frac{i}{2}:\bar{\psi}^{\mathrm{B}}\overleftrightarrow{\partial}_\theta\psi^{\mathrm{B}}:-\frac{i}{2}\partial_\theta T^{\mathrm{B}}+d_0T^{\mathrm{F}}$$

$$+(\bar{d}_0^{\mathrm{aux}}+i\bar{D}_0^{\mathrm{aux}}\partial_\theta)(T^{\mathrm{F}}+T^{\mathrm{B}})+\varepsilon(d_0,\bar{D}_0^{\mathrm{aux}}) \tag{7.2.10a}$$

$$T_{\mathrm{boost}}(D)=T^{\mathrm{F}}-2\bar{D}_0^{\mathrm{aux}}(T^{\mathrm{F}}+T^{\mathrm{B}})+\eta(d_0,\bar{D}_0^{\mathrm{aux}}),\qquad G_{\mathrm{boost}}^{+}(D)=\sqrt{2}\,e^{i\theta d_0}\bar{\psi}^{\mathrm{F}}\psi^{\mathrm{B}} \tag{7.2.10b}$$

$$G_{\mathrm{boost}}^{-}(D)=-\sqrt{2}\,ie^{-i\theta d_0}(\partial_\theta\bar{\psi}^{\mathrm{B}})\psi^{\mathrm{F}}+\sqrt{2}\,(\bar{d}_0^{\mathrm{aux}}+2i\bar{D}_0^{\mathrm{aux}}\partial_\theta)(e^{-i\theta d_0}\bar{\psi}^{\mathrm{B}}\psi^{\mathrm{F}}) \tag{7.2.10c}$$

$$\varepsilon(D)=\tfrac{1}{2}d_0^2+d_0\bar{D}_0^{\mathrm{aux}}-\tfrac{1}{2}\bar{D}_0^{\mathrm{aux}},\qquad \eta(D)=d_0-2d_0\bar{D}_0^{\mathrm{aux}}+\bar{d}_0^{\mathrm{aux}} \tag{7.2.10d}$$

$$c(\bar{D}_0^{\mathrm{aux}})=3(1-4\bar{D}_0^{\mathrm{aux}})=3k(\bar{D}_0^{\mathrm{aux}}) \tag{7.2.10e}$$

in terms of antiperiodic[15] complex Fermi and Bose quarks [2] $\psi^{\mathrm{F,B}}$ with $T^{\mathrm{F,B}}=:\bar{\psi}^{\mathrm{F,B}}\psi^{\mathrm{F,B}}:$. The representation of the auxiliary algebra (7.2.2) corresponding to the construction (7.2.10) is

$$\psi^{\mathrm{aux}}=\bar{\psi}^{\mathrm{B}}\psi^{\mathrm{F}},\qquad T^{\mathrm{aux}}=T^{\mathrm{F}}+T^{\mathrm{B}},\qquad \hat{k}=0,\quad \lambda=1 \tag{7.2.11}$$

so ψ^{aux} is not canonical and T^{aux} is not a tensor at the origin,[16] while $\bar{\psi}^{\mathrm{aux}}$ and $\bar{T}^{\mathrm{aux}}$ decouple. The weights and modeings of the boosted quarks are

$$h(\bar{\psi}_{\mathrm{boost}}^{\mathrm{F}})=h(b)=1-h(\psi_{\mathrm{boost}}^{\mathrm{F}})=\tfrac{1}{2}+\bar{D}_0^{\mathrm{aux}} \tag{7.2.12a}$$

$$\sigma(\bar{\psi}_{\mathrm{boost}}^{\mathrm{F}})=\sigma(b)=1-\sigma(\psi_{\mathrm{boost}}^{\mathrm{F}})=\tfrac{1}{2}-(d_0+\bar{d}_0^{\mathrm{aux}}) \tag{7.2.12b}$$

$$h(\bar{\psi}_{\mathrm{boost}}^{\mathrm{B}})=h(\beta)=1-h(\psi_{\mathrm{boost}}^{\mathrm{B}})=\bar{D}_0^{\mathrm{aux}} \tag{7.2.12c}$$

$$\sigma(\bar{\psi}_{\mathrm{boost}}^{\mathrm{B}})=\sigma(\beta)=1-\sigma(\psi_{\mathrm{boost}}^{\mathrm{B}})=\tfrac{1}{2}-\bar{d}_0^{\mathrm{aux}} \tag{7.2.12d}$$

[15] The equivalent form of (7.2.10) in terms of boosted quarks with arbitrary modeing is obtained with $\psi\to\psi_{\mathrm{boost}}$ and $d_0=\bar{d}_0^{\mathrm{aux}}=0$.

[16] In fact, $T_{\mathrm{boost}}^{\mathrm{aux}}(d_0)=T_m^{\mathrm{aux}}+d_0\delta_{m,0}$ is never a tensor of $L_m(D)$ since $\lambda+2\hat{k}D_0^{\mathrm{aux}}\neq 0$ for any value of the parameters in this deformation.

and the known $N=2$ superconformal ghost system is located at the points $\bar{D}_0^{\text{aux}}=\frac{3}{2}=d_0+\bar{d}_0^{\text{aux}}$ with $d_0=\frac{1}{2}(\text{NS}), 1(R)$.

It is amusing to pretend that the continuous $N=2$ superconformal construction (7.2.10) is the physical ghost system in a chiral sector of a D-dimensional superstring [4, 6, 53, 21, 17, 28]. The requirement $d_0+\bar{d}_0^{\text{aux}}=\bar{D}_0^{\text{aux}}$ maintains the $|SL_2\rangle_0=|0\rangle_{\text{BH}}$ state of the reparametrization ghosts throughout the deformation, and the further requirement $\bar{D}_0^{\text{aux}}=m+\frac{1}{2}$, $m\in\mathbb{Z}$, fixes integer-moding for these ghosts.[17] The resulting total central charge for matter plus ghosts in D dimensions is $3(D/2-1-4m)$ so the Weyl anomaly cancels for those dimensions

$$D=8m+2 \tag{7.2.13}$$

which allow a spacetime Majorana–Weyl fermion. The ghost weights $h(\bar{\psi}_{\text{boost}}^{\text{F}})=h(b)=(D+6)/8$, $h(\bar{\psi}_{\text{boost}}^{\text{B}})=h(\beta)=(D+2)/8$ follow with (7.2.12a, c) and (7.2.13).

8. Linear-Loaded Deformations and Spontaneous Breakdown

8.1. *c-Fixed Chiral Deformations*

Begin at the origin with a Sugawara construction $L_m[0]$ for level $2k/\psi^2$ of g, whose CSA currents T_0^a are defined on a base-lattice which is a sublattice with basis vectors $\mu(0)$ of the weight-lattice of g. The simplest non-flat deformation is the case of c-fixed linear-loading[18]

$$L_m(d_0(T_0))=L_m[0]+d_0^a(T_0)T_m^a+\tfrac{1}{2}kd_0^2(T_0)\delta_{m,0}, \qquad d_0^a(T_0)=e^a+f^{ab}T_0^b \tag{8.1.1}$$

characterized by linear-dependence of the deformation d_0 on the currents T_0. The present section treats only this linear-loaded case, which includes spontaneous breakdown [24, 25], while more general non-linear deformations are discussed in Sections 9 and 10, and Appendix B.

The boosted CSA currents

$$T_{\text{boost}}^a(d_0(T_0))_m\equiv T_m^a(d_0(T_0))=T_m^a+k(e^a+f^{ab}T_0^b)\delta_{m,0} \tag{8.1.2}$$

remain $(1,0)$ operators throughout the loaded deformation and $T_0^a(d_0)$ defines a level-dependent target-lattice which is a linear transformation of the base-lattice. The boosting of the charged operators is representation-dependent, and the modeing of these operators for arbitrary level of g at the origin is computed in Section 10. Although further analysis can be given with the constructions in (6.3.11b), (6.3.12b), and [54, 37, 48], we discuss here only level-one of simply laced g.

[17] The additional requirement $\bar{d}_0^{\text{aux}}=\bar{D}_0^{\text{aux}}-\frac{1}{2}$ (which implies $d_0=\frac{1}{2}$ with the previous requirements) maintains the $|SL_2\rangle_0=|0\rangle_{\text{BH}}^{\text{F}}\otimes|0\rangle_{\text{BH}}^{\text{B}}$ state of the NS sector, and the R sector is similarly constructed.

[18] The deformation (8.1.1) is $|SL_2\rangle_0$-preserving when $e=0$.

In this case, the Sugawara construction at the origin is equal to the construction on the maximal torus of g [12, 15, 18] so the loaded deformation (8.1.1) takes the simple form

$$L_m(d_0(T_0)) = \frac{1}{2}\sum_n \,{}^\times_\times\, T^a_{m+n}(d_0(T_0))\,T^a_{-n}(d_0(T_0))\,{}^\times_\times = L_m[0]\,|_{T_0 \to T_0(d_0(T_0))} \tag{8.1.3}$$

and the physics of the level-one deformation is entirely in the target-lattice. The vertex-operator construction [12–14] for non-zero weight $\mu(0)$ of representation R at the origin boosts according to[19]

$$(L_m(d_0(T_0)), R^{\mu(d_0)}_{\text{boost}}(z)) = z^m(z\partial_z + \tfrac{1}{2}m\mu^2(d_0))\,R^{\mu(d_0)}_{\text{boost}}(z) \tag{8.1.4a}$$

$$R^{\mu(d_0)}_{\text{boost}}(z) = c_{\mu(0)}(T_0)\,U^{\mu(d_0)}_{\text{boost}}(z), \qquad U^{\mu(d_0)}_{\text{boost}}(z) = \Gamma^0_{\mu(0)}(d_0, z)\,\Gamma^-_{\mu(d_0)}(z)\,\Gamma^+_{\mu(d_0)}(z), \tag{8.1.4b}$$

where $c_{\mu(0)}$ is the Klein transformation [46, 12–14] at the origin. The zero-mode factor is

$$\Gamma^0_{\mu(0)}(d_0, z) = e^{i\mu(0)\cdot q(0)}z^{-\sigma_{\mu(0)}(T_0(d_0))} \tag{8.1.5a}$$

$$= e^{i\mu(d_0)\cdot(q(d_0) - i\ln z\, T_0(d_0))} \tag{8.1.5b}$$

with

$$\mu(d_0) \equiv (1+f)\,\mu(0), \qquad \sigma_{\mu(0)}(T_0(d_0)) \equiv -\tfrac{1}{2}\mu^2(d_0) - \mu(d_0)\cdot T_0(d_0) \tag{8.1.6a}$$

$$q(d_0) \equiv q(0)(1+f)^{-1}, \qquad (q^a(d_0), T^b_m(d_0(T_0))) = i\delta^{ab}\delta_{m,0} \tag{8.1.6b}$$

and the higher-mode factors have the usual form $\Gamma^{\pm}_{\mu(d_0)} = \exp(i\mu(d_0)\cdot Q^{\pm})$.

The base-lattice translation-factor in (8.1.5a) maintains the algebra of the Klein transformation with the boosted vertex-operators throughout the deformation. Moreover, the forms of the zero-mode factor in (8.1.5a), (8.1.5b) show that the boosted vertex-operators translate from point to point on both the base-lattice and the target-lattice, whose basis vectors are $\mu(d_0)$.

The continuous shift of conformal-weight $h(\mu(d_0)) = \frac{1}{2}\mu^2(d_0)$ in (8.1.4a) corresponds to wavefunction renormalization in Thirring models [12], which is called change of compactification radius in Thirring strings [55–57]. Addition of a flat c-changing deformation $\{D^a_0 \neq 0\}$ does not modify the boosted vertex-operator constructions (8.1.4b) but further shifts their conformal-weight to

$$h(\mu(d_0), D_0) = \tfrac{1}{2}\mu^2(d_0) + \mu(d_0)\cdot D_0. \tag{8.1.7}$$

We also remark that linear-loaded deformation generically involves non-local distortion of the current algebra and other algebraic structure since $\mu(d_0)\cdot\mu'(d_0)$ is

[19] The flat-deformation limit $f \to 0$ of the boosted vertex-operators (8.1.4b) is the expected twist $\exp(i\theta e\cdot\mu)R^{\mu}(\theta)$ by the lattice translation-vector $d^a_0 = e^a$.

not generally an integer. Moreover, the modeing (8.1.6a) of the boosted tensors typically varies across the target-lattice, so that

$$(L_0(d_0(T_0)), R^{\mu(d_0)}_{\text{boost},\, m+\sigma_{\mu(0)}(T_0(d_0))}) = -R^{\mu(d_0)}_{\text{boost},m+\sigma_{\mu(0)}(T_0(d_0))}[m+\sigma_{\mu(0)}(T_0(d_0))] \quad (8.1.8a)$$

$$= [-m+\sigma_{-\mu(0)}(T_0(d_0))]\, R^{\mu(d_0)}_{\text{boost},m+\sigma_{\mu(0)}(T_0(d_0))} \quad (8.1.8b)$$

$$R^{\mu(d_0)}_{\text{boost},m+\sigma_{\mu(0)}(T_0(d_0))} \equiv \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} R^{\mu(d_0)}_{\text{boost}}(\theta) e^{i\theta[m+\sigma_{\mu(0)}(T_0(d_0))]} \quad (8.1.8c)$$

which complicates ordinary commutator and anticommutator structure on states.

8.2. *Restriction by $N=1$ World-Sheet* SUSY

Maintenance of $N=1$ world-sheet SUSY [4, 6], to prevent a flood of negative-norm states in the theory, is a serious restriction on linear-loaded deformations, as illustrated in the following simple model.

Begin by compactifying $10-d=2M$ dimensions of a conventional $(1, 0)$ supersymmetry at the origin, and suppressing the d physical dimensions, since they play no role in the analysis. The fermionized form of the internal contribution is the non-linear SUSY of Section 7.1 with $g=SU(2)^{2M}$, but analysis of linear deformations requires the bosonized form of the construction, with $3M$ bosonic fields $Q^I, I=1, ..., 3M$. The CSA currents T^{B} and hidden CSA currents T^{F} of the non-linear representation

$$T^{\text{B},a} = \partial_\theta Q^{M+a}, \qquad a=1, ..., 2M; \qquad T^{\text{F},b} = \partial_\theta Q^b, \quad b=1, ..., M \quad (8.2.1)$$

are defined on B = bosonic and F = fermionic sublattices, respectively. Then the supercharge at the origin

$$G[0] = \frac{1}{\sqrt{2}} \sum_{b=1}^{M} \xi_b U^{-\mu_b(0)}(T^{\text{B},2b-1} - iT^{\text{B},2b}) + \text{h.c.} \quad (8.2.2a)$$

$$\mu_b^I(0) = \delta_{I,b}, \qquad I, b=1, ..., 3M \quad (8.2.2b)$$

is obtained with the vertex-operator construction of fermions [12, 13, 15]

$$\psi^b = \xi_b U^{-\mu_b(0)}, \qquad \xi_b = \exp\left[i\pi \sum_{I=1}^{b-1} \text{int}(T_0^{\text{F},I})\right], \qquad b=1, ..., M \quad (8.2.3)$$

whose real and imaginary parts are the Cartan fermions S^a of the non-linear construction.

Our study of linearly loaded superconformal deformations follows the trial deformation method of Section 7. The starting point is the most general linear-loaded c-fixed conformal deformation

$$L_m(d_0(T_0)) = L_m[0] + (e_{\mathrm{B}}^a + f_{\mathrm{BB}}^{ab} T_0^{\mathrm{B},b} + f_{\mathrm{BF}}^{ab} T_0^{\mathrm{F},b}) T_m^{\mathrm{B},a} + (e_{\mathrm{F}}^a + f_{\mathrm{FB}}^{ab} T_0^{\mathrm{B},b} + f_{\mathrm{FF}}^{ab} T_0^{\mathrm{F},b}) T_m^{\mathrm{F},a} + \varepsilon(d_0(T_0))\delta_{m,0} \quad (8.2.4a)$$

$$T_m^{\mathrm{B},a}(d_0) = T_m^{\mathrm{B},a} + (e_{\mathrm{B}}^a + f_{\mathrm{BB}}^{ab} T_0^{\mathrm{B},b} + f_{\mathrm{BF}}^{ab} T_0^{\mathrm{F},b})\delta_{m,0} \quad (8.2.4b)$$

$$T_m^{\mathrm{F},a}(d_0) = T_m^{\mathrm{F},a} + (e_{\mathrm{F}}^a + f_{\mathrm{FB}}^{ab} T_0^{\mathrm{B},b} + f_{\mathrm{FF}}^{ab} T_0^{\mathrm{F},b})\delta_{m,0} \quad (8.2.4c)$$

with respect to the complete set of CSA and hidden CSA currents. The explicit weights and modeings of the boosted "fermions" $\psi_{\mathrm{boost}}^b(d_0) = \xi_b U_{\mathrm{boost}}^{-\mu_b(d_0)}$ are

$$\mu_b^I(d_0) = \begin{cases} (1 + f_{\mathrm{FF}})^{Ib}, & 1 \leqslant I \leqslant M \\ f_{\mathrm{BF}}^{I-m,b}, & M+1 \leqslant I \leqslant 3M \end{cases} \quad (8.2.5a)$$

$$\sigma_b(T_0(d_0)) = -\tfrac{1}{2}\mu_b^2(d_0) + \{[(1 + f_{\mathrm{FF}})^{\mathrm{T}}(1 + f_{\mathrm{FF}}) + f_{\mathrm{BF}}^{\mathrm{T}} f_{\mathrm{BF}}] T_0^{\mathrm{F}} + [(1 + f_{\mathrm{FF}})^{\mathrm{T}} f_{\mathrm{FB}} + f_{\mathrm{BF}}^{\mathrm{T}}(1 + f_{\mathrm{BB}})] T_0^{\mathrm{B}} + (1 + f_{\mathrm{FF}})^{\mathrm{T}} e_{\mathrm{F}} + f_{\mathrm{BF}}^{\mathrm{T}} e_{\mathrm{B}}\}^b, \quad (8.2.5b)$$

where T = transpose, and the modeing of the boosted antifermions is obtained with the opposite sign of the bracket in (8.2.5b).

The (trial) boosted supercharge is constructed by boosting the components of $G[0]$ and allowing a shift. Full normal-ordering is required because (b not summed)

$$(T_m^{\mathrm{B},a}(d_0), \psi_{\mathrm{boost}}^b(d_0)) = -f_{\mathrm{BF}}^{ab} e^{im\theta} \psi_{\mathrm{boost}}^b(d_0) \quad (8.2.6)$$

and the result

$$G_{\mathrm{boost}}(d_0(T_0)) = \frac{1}{\sqrt{2}} \sum_{b=1}^{M} \left\{ :\psi_{\mathrm{boost}}^b(d_0)(T^{\mathrm{B},2b-1}(d_0) - iT^{\mathrm{B},b}(d_0)): - (f_{\mathrm{BF}}^{2b-1,b} - if_{\mathrm{BF}}^{2b,b}) \left(i\partial_\theta + \frac{1}{2}\right) \psi_{\mathrm{boost}}^b(d_0) \right\} + \text{h.c.} \quad (8.2.7)$$

transforms with weight $\frac{3}{2}$ when $\mu_b^2(d_0) = 1$.

Maintenance of the full $N=1$ SUSY proceeds thru a sequence of further requirements on G_{boost} in a neighborhood of the origin:

(1) *Locality of* $(G, G)_+$. The stronger condition $\mu_a(d_0) \cdot \mu_b(d_0) = \mu_a(0) \cdot \mu_b(0) = \delta_{ab}$, or

$$(1 + f_{\mathrm{FF}})^{\mathrm{T}}(1 + f_{\mathrm{FF}}) + f_{\mathrm{BF}}^{\mathrm{T}} f_{\mathrm{BF}} = 1, \quad (8.2.8)$$

is necessary for mutual locality of the boosted fermions, and hence locality of G with itself.

(2) *Homogeneity of G.* $T^{\mathrm{B}}(d_0)$ in (8.2.7) remains integer-moded and any shift

of the modeing (8.2.5b) is opposite for the fermionic and antifermionic components of G, so we must require $\sigma_b(T_0(d_0)) = \sigma_b(T_0) = -\frac{1}{2} + T_0^{\mathrm{F},b}$, or

$$(1+f_{\mathrm{FF}})^{\mathrm{T}} f_{\mathrm{FB}} + f_{\mathrm{BF}}^{\mathrm{T}}(1+f_{\mathrm{BB}}) = (1+f_{\mathrm{FF}})^{\mathrm{T}} e_{\mathrm{F}} + f_{\mathrm{BF}}^{\mathrm{T}} e_{\mathrm{B}} = 0. \tag{8.2.9}$$

The usual local anticomutation relations for $\psi_{\mathrm{boost}}(\theta)$, $\bar{\psi}_{\mathrm{boost}}(\theta')$ are recovered at this point.

(3) *G closes to L.* Direct computation of $(G, G)_+$ by operator product expansion exhibits several classes of spurious terms related to (8.2.6) whose elimination requires

$$f_{\mathrm{BF}} = 0 \tag{8.2.10}$$

and then the algebra of all components $\psi_{\mathrm{boost}}(d_0)$, $\bar{\psi}_{\mathrm{boost}}(d_0)$, $T^{\mathrm{B}}(d_0)$ is normal. The overall solution

$$1 + f_{\mathrm{FF}} = \text{orthogonal}, \qquad f_{\mathrm{FB}} = e_{\mathrm{F}} = 0 \tag{8.2.11}$$

follows with (8.2.8)–(8.2.10).

Our conclusion is that c-fixed continuously deformed $N=1$ world-sheet SUSY is only possible on the target-lattice

$$T_0^{\mathrm{B}}(d_0) = (1+f_{\mathrm{BB}}) T_0^{\mathrm{B}} + e_{\mathrm{B}} \tag{8.2.12a}$$

$$T_0^{\mathrm{F}}(d_0) = (1+f_{\mathrm{FF}}) T_0^{\mathrm{F}} = (\text{orthogonal}) \cdot T_0^{\mathrm{F}}. \tag{8.2.12b}$$

The orthogonal transformation (8.2.12b) on the fermionic lattice does not affect the eigenvalues of $L_0(d_0)$ and the linear transformation (8.2.12a) of the bosonic lattice is of the Narain type [24], which is known not to spontaneously break spacetime SUSY.

Although our argument required only $(1,0)$ superconformal invariance, our conclusion is consistent with that of [25]. After completion of this work we learned that more general models have been discussed along these lines in [58].

8.3. *Conformal Field Theory and $SO_x(p,q)$*

Define a c-fixed linear-loaded conformal field theory for levels $2k/\psi^2$, $2\bar{k}/\bar{\psi}^2$ of Lie algebra $g \oplus \bar{g}$ at the origin by the doubling [59]

$$L_m(d_0(T_0, \bar{T}_0)) = L_m[0] + d_0^a(T_0, \bar{T}_0) T_m^a + \tfrac{1}{2} k d_0^2(T_0, \bar{T}_0) \delta_{m,0}, \qquad a = 1, \ldots, p \tag{8.3.1a}$$

$$\bar{L}_m(\bar{d}_0(\bar{T}_0, T_0)) = \bar{L}_m[0] + \bar{d}_0^{\bar{a}}(\bar{T}_0, T_0) \bar{T}_m^{\bar{a}} + \tfrac{1}{2} \bar{k} \bar{d}_0^2(\bar{T}_0, T_0) \delta_{m,0}, \qquad \bar{a} = 1, \ldots, q \tag{8.3.1b}$$

$$d_0^a(T_0, \bar{T}_0) = e^a(\bar{T}_0) + f^{ab}(\bar{T}_0) T_0^b, \qquad \bar{d}_0^{\bar{a}}(\bar{T}_0, T_0) = \bar{e}^{\bar{a}}(T_0) + \bar{f}^{\bar{a}\bar{b}}(T_0) \bar{T}_0^{\bar{b}} \tag{8.3.1c}$$

and the level-matching condition below. The boosted CSA currents

$$\begin{pmatrix} T^a_m(d_0(T_0, \bar{T}_0)) \\ \bar{T}^{\bar{a}}_m(\bar{d}_0(\bar{T}_0, T_0)) \end{pmatrix} = \begin{pmatrix} T^a_m + k(e^a(\bar{T}_0) + f^{ab}(\bar{T}_0) T^b_0)\delta_{m,0} \\ \bar{T}^{\bar{a}}_m + \bar{k}(\bar{e}^{\bar{a}}(T_0) + \bar{f}^{\bar{a}b}(T_0)\bar{T}^b_0)\delta_{m,0} \end{pmatrix} \tag{8.3.2}$$

remain $(1, 0)$ and $(0, 1)$ tensors, respectively, throughout the deformation, and $(T_0(d_0), \bar{T}_0(\bar{d}_0))$ defines a $(p+q)$-dimensional level-dependent target-lattice. The level-matching condition

$$L_0(d_0(T_0, \bar{T}_0)) = \bar{L}_0(\bar{d}_0(\bar{T}_0, T_0)) \tag{8.3.3}$$

decomposes into the condition at the origin $L_0[0] = \bar{L}_0[0]$ and a restriction on the deformations

$$T_0^2(d_0(T_0, \bar{T}_0)) - x^2\bar{T}_0^2(\bar{d}_0(\bar{T}_0, T_0)) = T_0^2 - x^2\bar{T}_0^2, \qquad x = \sqrt{k/\bar{k}}. \tag{8.3.4}$$

The restriction defines an $SO_x(p, q)$ family of target-lattices

$$\begin{pmatrix} T_0(d_0) \\ \bar{T}_0(\bar{d}_0) \end{pmatrix} = SO_x(p, q) \cdot \begin{pmatrix} T_0 \\ \bar{T}_0 \end{pmatrix} \tag{8.3.5a}$$

$$SO_x(p, q) \equiv K_x^{-1} SO(p, q) K_x, \qquad K_x = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & x\mathbb{1}_q \end{pmatrix}, \tag{8.3.5b}$$

where the group $SO_x(p, q)$ is isomorphic to $SO(p, q)$ by the similarity transformation K_x. A consequence of (8.3.5) is that $e^a(0) = \bar{e}^{\bar{a}}(0) = 0$, so this class of deformations is automatically $|SL_2\rangle_0$-preserving (see also Section 10).

The simplest example is the $SO_x(1, 1)$ deformation for arbitrary levels of $g = \bar{g} = SU(2)$,

$$\begin{pmatrix} T_0(d_0) \\ \bar{T}_0(\bar{d}_0) \end{pmatrix} = \begin{pmatrix} \cosh\alpha & x\sinh\alpha \\ x^{-1}\sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} T_0 \\ \bar{T}_0 \end{pmatrix} \tag{8.3.6a}$$

$$\begin{pmatrix} e(\bar{T}_0) \\ \bar{e}(T_0) \end{pmatrix} = \begin{pmatrix} \bar{T}_0 \\ T_0 \end{pmatrix} \frac{x}{k} \sinh\alpha, \qquad \begin{pmatrix} f(\bar{T}_0) \\ \bar{f}(T_0) \end{pmatrix} = \begin{pmatrix} 1 \\ x^2 \end{pmatrix} \frac{1}{k}(\cosh\alpha - 1), \tag{8.3.6b}$$

which breaks the group to $U(1) \otimes U(1)$ and corresponds to continuous compactification radius $r(d_0) = \sqrt{2}\exp(-\alpha)/\mu(0)$ when $k = \bar{k} = 1$ [60].

For level-one of simply laced g and $\bar{g}$ the Sugawara construction equals the construction on the maximal torus of $g \oplus \bar{g}$ [12, 15, 18],

$$\begin{aligned} L_m(d_0(T_0, \bar{T}_0)) &= L_m[0] \,|_{T_0 \to T_0(d_0(T_0, \bar{T}_0))}, \\ \bar{L}_m(\bar{d}_0(\bar{T}_0, T_0)) &= \bar{L}_m[0] \,|_{\bar{T}_0 \to \bar{T}_0(\bar{d}_0(\bar{T}_0, T_0))}, \end{aligned} \tag{8.3.7}$$

so the effective flat directions fill $SO(p, q)/SO(p) \otimes SO(q)$ in this case [24]. The further requirement of full modular invariance [23] apparently restricts the base-lattice to even self-dual [24] and shifted odd self-dual [25] lattices. Finally, the boosted vertex-operators of the level-one closed string [59]

$$U_{\text{boost}}^{\mu(d_0),\bar{\mu}(\bar{d}_0)}(z,\bar{z}) = U_{\text{boost}}^{\mu(d_0)}(z)\,\bar{U}_{\text{boost}}^{\bar{\mu}(\bar{d}_0)}(\bar{z}) \tag{8.3.8a}$$

$$= \Gamma^0_{\mu(0)}(d_0, z)\,\bar{\Gamma}^0_{\bar{\mu}(0)}(\bar{d}_0, \bar{z})\,\Gamma^-_{\mu(d_0)}(z)\,\bar{\Gamma}^-_{\bar{\mu}(\bar{d}_0)}(\bar{z})\,\Gamma^+_{\mu(d_0)}(z)\,\bar{\Gamma}^+_{\bar{\mu}(\bar{d}_0)}(\bar{z}) \tag{8.3.8b}$$

$$\begin{aligned} \Gamma^0_{\mu(0)}(d_0, z) &= e^{i\mu(0)\cdot q(0)} z^{(1/2)\mu^2(d_0) + \mu(d_0)\cdot T_0(d_0)}, \\ \bar{\Gamma}^0_{\bar{\mu}(0)}(\bar{d}_0, \bar{z}) &= e^{i\bar{\mu}(0)\cdot \bar{q}(0)} \bar{z}^{(1/2)\bar{\mu}^2(\bar{d}_0) + \bar{\mu}(\bar{d}_0)\cdot \bar{T}_0(\bar{d}_0)} \end{aligned} \tag{8.3.8c}$$

$$\begin{pmatrix} \mu(d_0) \\ \bar{\mu}(\bar{d}_0) \end{pmatrix} = \begin{pmatrix} (1+f(\bar{T}_0))\mu(0) \\ (1+\bar{f}(T_0))\bar{\mu}(0) \end{pmatrix} \tag{8.3.8d}$$

map from point to point on the base- and target-lattices and transform as $(\frac{1}{2}\mu^2(d_0), \frac{1}{2}\bar{\mu}^2(\bar{d}_0))$ as expected.

9. c-Fixed Non-linear Deformations

9.1. *Vertex-Operators for Arbitrarily Deformed Lattices*

This section discusses the arbitrarily loaded c-fixed deformation

$$L_m(d_0(T_0)) = L_m[0] + d_0^a(T_0)T_m^a + \tfrac{1}{2}kd_0^2(T_0)\delta_{m,0}, \qquad d_0^a(T_0) = \text{arbitrary} \tag{9.1.1}$$

reserving off-CSA rotation for Appendix B and c-changing generalization for Section 10. The boosted CSA currents

$$T^a_{\text{boost}}(d_0(T_0))_m \equiv T^a_m(d_0(T_0)) = T^a_m + kd_0^a(T_0)\delta_{m,0} \tag{9.1.2}$$

are $(1, 0)$ throughout the deformation and $T_0(d_0) = T_0 + kd_0(T_0)$ defines a level-dependent arbitrarily deformed target-lattice as a map from the base-lattice with basis vectors $\mu(0)$.

The *local* basis on the target-lattice

$$\mu_i(T_0(d_0), \mu_i(0)) \equiv T_0(d_0)\,|_{T_0 \to T_0 + \mu_i(0)} - T_0(d_0) \tag{9.1.3}$$

is the set of (one-step) translation vectors at the point $T_0(d_0)$. It follows from (9.1.3) that translations on the target-lattice form a group, since multi-step translations are additive,

$$\mu(T_0(d_0), \mu_i(0) + \mu_j(0)) = \mu_i(T_0(d_0), \mu_i(0)) + \mu_j(T_0(d_0) + \mu_i(T_0(d_0), \mu_i(0)), \mu_j(0)), \tag{9.1.4}$$

and one-step translations are invertible,

$$-\mu_i(T_0(d_0), \mu_i(0)) = \mu_i(T_0(d_0) + \mu_i(T_0(d_0), \mu_i(0)), -\mu_i(0)), \tag{9.1.5}$$

which follows from (9.1.4) with $\mu_j(0) = -\mu_i(0)$. Moreover, the local basis reduces to the constant basis (8.1.6a) when the deformation is linear.

The deformation terms in $L_0(d_0)$ and $L_{-1}(d_0)$ of (9.1.1) are proportional to $d_0(T_0)$, while any highest-weight- and SL_2-state $|SL_2\rangle_0$ at the origin has $T_0^a = 0$ (see Section 6.3). It follows that any deformation with

$$d_0(0) = 0 \tag{9.1.6}$$

is $|SL_2\rangle_0$-preserving (see also Section 10).

The boosting of the charged operators (see Section 10) is representation-dependent, and we discuss only level-one of simply laced g at the origin, for which

$$L_m(d_0(T_0)) = L_m[0]\,|_{T_0 \to T_0(d_0(T_0))} \tag{9.1.7}$$

follows with the steps [12, 15, 18] of (8.1.3). Then the boosted vertex-operators of the arbitrarily deformed lattice

$$R_{\rm boost}^{\mu(T_0(d_0),\mu(0))}(z) = c_{\mu(0)}(T_0)\, U_{\rm boost}^{\mu(T_0(d_0),\mu(0))}(z) \tag{9.1.8a}$$

$$U_{\rm boost}^{\mu(T_0(d_0),\mu(0))}(z) = \Gamma^0_{\mu(0)}(T_0(d_0), z)\, \Gamma^-_{\mu(T_0(d_0),\mu(0))}(z)\, \Gamma^+_{\mu(T_0(d_0),\mu(0))}(z) \tag{9.1.8b}$$

$$\Gamma^0_{\mu(0)}(T_0(d_0), z) = e^{i\mu(0)\cdot q(0)} z^{-\sigma_{\mu(0)}(T_0(d_0))} \tag{9.1.8c}$$

$$\sigma_{\mu(0)}(T_0(d_0)) = -\tfrac{1}{2}\mu^2(T_0(d_0), \mu(0)) - \mu(T_0(d_0), \mu(0)) \cdot T_0(d_0) \tag{9.1.8d}$$

are constructed in terms of the local basis (9.1.3).

The relations

$$T_0(d_0)\, R_{\rm boost}^{\mu(T_0(d_0),\mu(0))}(z) = R_{\rm boost}^{\mu(T_0(d_0),\mu(0))}(z)[T_0(d_0) + \mu(T_0(d_0), \mu(0))] \tag{9.1.9a}$$

$$T_0\, R_{\rm boost}^{\mu(T_0(d_0),\mu(0))}(z) = R_{\rm boost}^{\mu(T_0(d_0),\mu(0))}(z)(T_0 + \mu(0)) \tag{9.1.9b}$$

show that the boosted vertex-operators simultaneously generate the local basis on the target-lattice and the constant basis on the base-lattice, although there is no simple way to express the zero-mode factor (9.1.8c) entirely in terms of quantities on the target-lattice. Moreover, since the conformal-weight $h \sim \mu^2/2$ of a vertex-operator is a distance to the next lattice point, it is not surprising that the operators transform as *local* tensors

$$(L_m(d_0(T_0)), R_{\rm boost}^{\mu(T_0(d_0),\mu(0))}(z)) = R_{\rm boost}^{\mu(T_0(d_0),\mu(0))}(z)(\overleftarrow{\partial}_z z + m h_{\rm R}) z^m \tag{9.1.10a}$$

$$= z^m(z\partial_z + m h_{\rm L})\, R_{\rm boost}^{\mu(T_0(d_0),\mu(0))}(z) \tag{9.1.10b}$$

$$h_{\rm R} = \tfrac{1}{2}\mu^2(T_0(d_0), \mu(0)), \qquad h_{\rm L} = \tfrac{1}{2}\mu^2(T_0(d_0), -\mu(0)) \tag{9.1.10c}$$

with lattice- and direction-dependent conformal-weight.

9.2. *Conformal Field Theory and Local $SO_x(p, q)$*

The c-fixed non-linear loaded conformal field theory for level $2k/\psi^2$, $2\bar{k}/\bar{\psi}^2$ of Lie algebra $g \oplus \bar{g}$ at the origin is

$$L_m(d_0(T_0, \bar{T}_0)) = L_m[0] + d_0^a(T_0, \bar{T}_0)T_m^a + \tfrac{1}{2}kd_0^2(T_0, \bar{T}_0)\delta_{m,0}, \qquad a = 1, \ldots, p \tag{9.2.1a}$$

$$\bar{L}_m(\bar{d}_0(\bar{T}_0, T_0)) = \bar{L}_m[0] + \bar{d}_0^{\bar{a}}(\bar{T}_0, T_0)\bar{T}_m^{\bar{a}} + \tfrac{1}{2}\bar{k}\bar{d}_0^2(\bar{T}_0, T_0)\delta_{m,0}, \qquad \bar{a} = 1, \ldots, q \tag{9.2.1b}$$

$$\begin{aligned} T_m^a(d_0(T_0, \bar{T}_0)) &= T_m^a + kd_0^a(T_0, \bar{T}_0)\delta_{m,0}, \\ \bar{T}_m^{\bar{a}}(\bar{d}_0(\bar{T}_0, T_0)) &= \bar{T}_m^{\bar{a}} + \bar{k}\bar{d}_0^{\bar{a}}(\bar{T}_0, T_0)\delta_{m,0}, \end{aligned} \tag{9.2.1c}$$

where $T(d_0(T_0, \bar{T}_0))$, $\bar{T}(\bar{d}_0(\bar{T}_0, T_0))$ are (1, 0), (0, 1) operators throughout the deformation, and the local basis for the target-lattice $(T_0(d_0(T_0, \bar{T}_0)), \bar{T}_0(\bar{d}_0(\bar{T}_0, T_0)))$ is

$$\begin{pmatrix} \mu(T_0(d_0), \mu(0)) \\ \bar{\mu}(\bar{T}_0(\bar{d}_0), \bar{\mu}(0)) \end{pmatrix} = \begin{pmatrix} T_0(d_0)\,|_{T_0 \to T_0 + \mu(0)} - T_0(d_0) \\ \bar{T}_0(\bar{d}_0)\,|_{\bar{T}_0 \to \bar{T}_0 + \bar{\mu}(0)} - \bar{T}_0(\bar{d}_0) \end{pmatrix}. \tag{9.2.2}$$

The level-matching condition $L_0(d_0(T_0, \bar{T}_0)) = \bar{L}_0(\bar{d}_0(\bar{T}_0, T_0))$ leads again to the restriction (8.3.4), which now defines a continuous *local* $SO_x(p, q)$ family of target-lattices

$$\begin{pmatrix} T_0(d_0) \\ \bar{T}_0(\bar{d}_0) \end{pmatrix} = \text{local } SO_x(p, q) \cdot \begin{pmatrix} T_0 \\ \bar{T}_0 \end{pmatrix}, \qquad \text{local } SO_x(p, q) \equiv K_x^{-1} \text{ local } SO(p, q) K_x \tag{9.2.3}$$

for arbitrary levels of $g \oplus \bar{g}$. The matrices of local $SO(p, q)$ are formed by taking arbitrary base-lattice dependence for the parameters of $SO(p, q)$, as in the local $SO_x(1, 1)$ example

$$\begin{pmatrix} d_0(T_0, \bar{T}_0) \\ \bar{d}_0(\bar{T}_0, T_0) \end{pmatrix} = \frac{1}{k} \begin{pmatrix} x\bar{T}_0 \sinh \alpha(T_0, \bar{T}_0) - T_0[1 - \cosh \alpha(T_0, \bar{T}_0)] \\ xT_0 \sinh \alpha(T_0, \bar{T}_0) - x^2\bar{T}_0[1 - \cosh \alpha(T_0, \bar{T}_0)] \end{pmatrix} \tag{9.2.4}$$

which corresponds to $\alpha \to \alpha(T_0, \bar{T}_0)$ in (8.3.6a).

The non-linear transformation (9.2.3) implies that $d_0(0, 0) = \bar{d}_0(0, 0) = 0$ when the parameters of local $SO_x(p, q)$ are well-behaved, so this class of deformations is automatically $|SL_2\rangle_0$-preserving (see Section 10).

For level-one of simply laced g and $\bar{g}$ at the origin the equivalence (8.3.7) is maintained, and the flat directions fill local $SO(p, q)$/local$\{SO(p) \otimes SO(q)\}$ in this case. The boosted vertex-operators $U_{\text{boost}}^{\mu, \bar{\mu}}$ which generate the local basis (9.2.2) are obtained by the substitution

$$\begin{pmatrix} \mu(d_0) \\ \bar{\mu}(\bar{d}_0) \end{pmatrix} \to \begin{pmatrix} \mu(T_0(d_0), \mu(0)) \\ \bar{\mu}(\bar{T}_0(\bar{d}_0), \bar{\mu}(0)) \end{pmatrix} \tag{9.2.5}$$

in (8.3.8a)–(8.3.8c) and transform as local tensors with $(h_{\mathrm{R}}, \bar{h}_{\mathrm{R}}) = (\frac{1}{2}\mu^2(T_0(d_0), \mu(0)), \frac{1}{2}\bar{\mu}^2(\bar{T}_0(\bar{d}_0), \bar{\mu}(0))$, as expected.

According to (8.3.4), the non-Gaussian lattice partition functions of the level-one non-linear deformations remain invariant under the modular subgroup $\tau \to \tau + 1$, so long as the theory at the origin was invariant, but use of these constructions as conformal building-blocks for fully modular-invariant strings is problematic.

10. General Loaded Deformations

The arbitrarily loaded deformation[20] $L_m[d(T_0), D_0(T_0)]$ in (2.11) is unitary-equivalent to the zero-mode deformation

$$L_m(d_0(T_0), D_0(T_0)) = L_m[0] + (d_0^a(T_0) + mD_0^a(T_0))\,T_m^a + \tfrac{1}{2}k(d_0^2(T_0) - D_0^2(T_0))\,\delta_{m,0} \tag{10.1a}$$

$$c(D_0(T_0)) = c(0) - 12kD_0^a(T_0)D_0^a(T_0) \tag{10.1b}$$

with $A(T_0) = \sum_{m \neq 0} d_{-m}^a(T_0)\,T_m^a/m$ as in Section 4.1, and the boosted CSA currents

$$T_m^a(d_0(T_0)) = T_m^a + kd_0^a(T_0)\,\delta_{m,0} \tag{10.2a}$$

$$(L_m(d_0(T_0), D_0(T_0)), T_n^a(d_0(T_0))) = -nT_{m+n}^a(d_0(T_0)) + km^2 D_0^a(T_0)\,\delta_{m,-n} \tag{10.2b}$$

are not tensors in the direction of $D_0(T_0)$, as expected. The effect of the operator central charge on boosted charged operators is discussed below.

The fixed-state phenomenon of Section 4.1 is generic: The states of each module at the origin are fixed eigenstates of T_0^a and $L_0(d_0(T_0), D_0(T_0))$ throughout the deformation, so the states of the deformed module are a continuous relabeling of the states at the origin. For application below we compute explicitly

$$|h, p\rangle: \quad h(d_0, D_0, p) = h + p \cdot d_0(p) + \tfrac{1}{2}k[d_0^2(p) - D_0^2(p)] \tag{10.3a}$$

$$T_{-m}^a\,|h, p\rangle: \quad h(d_0, D_0, p; a, m) = m + h + p \cdot d_0(p) + \tfrac{1}{2}k[d_0^2(p) - D_0^2(p)] \tag{10.3b}$$

$$E_{-m}^\alpha\,|h, p\rangle: \quad h(d_0, D_0, p; \alpha, m) = m + h + (p+\alpha) \cdot d_0(p+\alpha) + \tfrac{1}{2}k[d_0^2(p+\alpha) - D_0^2(p+\alpha)], \tag{10.3c}$$

where $|h, p\rangle$ is a highest-weight-state at the origin with $T_0^a = p^a$ and $m \geqslant 1$.

The fixed-state phenomenon also implies that a positive norm $\|\ \|_0^2$ at the origin is available throughout the c-changing deformation. As an illustration, consider the

[20] See also Appendix B.

fermionic constructions of generalized ghosts in Section 6.3, whose ψ, $\bar{\psi}$ norm[21] at the origin is positive. In this norm, for example,

$$\|L_{-m}(d_0, D_0)\,|h, p\rangle\|_0^2 \equiv \langle h, p|\, L^{\dagger}_{-m}(d_0, D_0) L_{-m}(d_0, D_0)\,|h, p\rangle \geqslant 0 \quad (10.4)$$

while the sign of $\langle h, p|\, L_m(d_0, D_0) L_{-m}(d_0, D_0)\,|h, p\rangle$ depends on $c(D_0)$ [41].

The general deformation (10.1a) is $|SL_2\rangle_0$-preserving when

$$d_0(0) = D_0(0) \quad (10.5)$$

which unifies the c-changing and c-fixed mechanisms (6.3.3) and (9.1.6).

The level-matching condition $L_0(d_0(T_0, \bar{T}_0), D_0(T_0, \bar{T}_0)) = \bar{L}_0(\bar{d}_0(\bar{T}_0, T_0), \bar{D}_0(\bar{T}_0, T_0))$ of the corresponding conformal field theory restricts the deformations according to

$$[T_0^2(d_0)/k + \bar{D}_0^2(\bar{T}_0, T_0)\bar{k}] - [\bar{T}_0^2(\bar{d}_0)/\bar{k} + D_0^2(T_0, \bar{T}_0)k] = T_0^2/k - \bar{T}_0^2/\bar{k} \quad (10.6)$$

which generalizes the $SO_x(p, q)$ condition (8.3.4) and interprets the c-changing deformations D_0, $\bar{D}_0$ as *auxiliary compactified spacetime dimensions*, doubling the original $(T_0, \bar{T}_0)$.

The particular solution[22] of (10.6)

$$\begin{pmatrix} T_0(d_0) \\ \bar{D}_0(\bar{T}_0, T_0) \\ \bar{T}_0(\bar{d}_0) \\ D_0(T_0, \bar{T}_0) \end{pmatrix} = \text{local } SO_{k,\bar{k}}(p+q, q\dot{+}p) \cdot \begin{pmatrix} T_0 \\ 0 \\ \bar{T}_0 \\ 0 \end{pmatrix},$$
$$K = \begin{pmatrix} \mathbb{1}_p & & & \\ & \sqrt{k\bar{k}}\,\mathbb{1}_q & & \\ & & \sqrt{k/\bar{k}}\,\mathbb{1}_q & \\ & & & k\,\mathbb{1}_p \end{pmatrix} \quad (10.7)$$

employs the matrices of local $SO_{k,\bar{k}}(p+q, q+p) \equiv K^{-1}$ local $SO(p+q, q+p)K$. In this case, the continuous family of target-lattices fills the coset-space

$$\text{local } SO_{k,\bar{k}}(p+q, q+p)/\text{local } SO_x(q, p) \quad (10.8)$$

[21] It also follows that the b, c norm with $L_m^{\dagger}(D) = L_{-m}(D)$ of footnote 13 is not positive at the origin ($c = 1$) since the norm is not positive for most $c < 1$ [41].

[22] Other solutions to (10.6) exist which are not automatically $|SL_2\rangle_0$-preserving, e.g., local $SO_x(p, q)$ on $(T_0, \bar{T}_0)$ and $\bar{D}_0^2(\bar{T}_0, T_0) = x^2 D_0^2(T_0, \bar{T}_0)$. The $|SL_2\rangle_0$-preserving restriction of this solution is contained in (10.7).

and the solution is automatically $|SL_2\rangle_0$-preserving since

$$d_0(0,0)=D_0(0,0)=\bar{d}_0(0,0)=\bar{D}_0(0,0)=0 \tag{10.9}$$

follows from (10.7) when the parameters of the deformation are well-behaved.

We finally remark on the construction of the boosted charged operators in the general case. The moding $\sigma_\alpha/\mathbb{Z}$ in the relations

$$\begin{aligned}(L_0(d_0(T_0), D_0(T_0)), E^{\alpha}_{\text{boost}}(d_0, D_0)_{m+\sigma_\alpha(T_0(d_0), D_0)})\\ = -E^{\alpha}_{\text{boost}}(d_0, D_0)_{m+\sigma_\alpha(T_0(d_0), D_0)}(m+\sigma_\alpha(T_0(d_0), D_0))\end{aligned} \tag{10.10a}$$

$$E^{\alpha}_{\text{boost}}(d_0, D_0)_{m+\sigma_\alpha(T_0(d_0), D_0)} \equiv \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} E^{\alpha}_{\text{boost}}(d_0, D_0, \theta) e^{i\theta(m+\sigma_\alpha(T_0(d_0), D_0))} \tag{10.10b}$$

$$\begin{aligned}\sigma_\alpha(T_0(d_0), D_0) = \frac{1}{k}\left(\frac{1}{2}\alpha^2 + \alpha\cdot T_0 - \frac{1}{2}\alpha^2(T_0(d_0), \alpha) - \alpha(T_0(d_0), \alpha)\cdot T_0(d_0)\right)\\ + \frac{1}{2}k[D_0^2(T_0+\alpha) - D_0^2(T_0)]\end{aligned} \tag{10.10c}$$

is computed from the energy gap in (10.3a), (10.3c) where $\alpha(T_0(d_0), \alpha)$ is the local basis (9.1.3) with $\mu(0)=\alpha$. The result (10.10c) agrees mod $\mathbb{Z}$ with the modeing of the local-tensor vertex-operators (9.1.8) when $k=1$ and $D_0=0$.

On the other hand, it follows from (A.3b) that a boosted representation R^{μ}_{boost} satisfies $(T^a_0, R^{\mu}_{\text{boost}})\neq 0$ for generic continuous deformation, and hence

$$(c(D_0(T_0)), R^{\mu}_{\text{boost}})\neq 0 \tag{10.11}$$

when the central charge is an operator. Then examination of the Jacobi identity among the operators $\{L_m(d_0, D_0), L_n(d_0, D_0), R^{\mu}_{\text{boost}}\}$ shows that R^{μ}_{boost} cannot be a tensor or even a local tensor in this case.

For example, construct general c-changing vertex-operators $R^{\mu(T_0(d_0), \mu(0))}_{\text{boost}}(D_0, z)$ for level-one of simply laced g at the origin by replacing $\sigma_{\mu(0)}(T_0(d_0))$ in (9.1.8a)–(9.1.8c) with

$$\begin{aligned}\sigma_{\mu(0)}(T_0(d_0), D_0) = -\tfrac{1}{2}\mu^2(T_0(d_0), \mu(0)) - \mu(T_0(d_0), \mu(0))\cdot T_0(d_0)\\ + \tfrac{1}{2}[D_0^2(T_0+\mu(0)) - D_0^2(T_0)]\end{aligned} \tag{10.12}$$

which agrees mod $\mathbb{Z}$ with (10.10c) when $\mu(0)=\alpha$, T_0 is on the root-lattice, and $k=1$. These vertex-operators transform as

$$(L_0(d_0(T_0), D_0(T_0)), R^{\mu(T_0(d_0), \mu(0))}_{\text{boost}}(D_0, z)) = z\partial_z R^{\mu(T_0(d_0), \mu(0))}_{\text{boost}}(D_0, z) \tag{10.13}$$

and generate the local basis (9.1.3) of the target-lattice, but the operators are not tensors in general since

$$(L_m(d_0(T_0), D_0(T_0)), R^{\mu(T_0(d_0),\mu(0))}_{\text{boost}}(D_0, z))$$
$$= R^{\mu(T_0(d_0),\mu(0))}_{\text{boost}}(D_0, z)\{[\tilde{\partial}_z z + m(\tfrac{1}{2}\mu^2(T_0(d_0), \mu(0)) + \mu(T_0(d_0), \mu(0)) \cdot D_0(T_0))$$
$$+ \tfrac{1}{2}(D_0^2(T_0 + \mu(0)) - D_0^2(T_0))] z^m + mT_m \cdot (D_0(T_0 + \mu(0)) - D_0(T_0))\} \qquad (10.14)$$

is verified for $m \neq 0$. The c-changing vertex-operators reduce to the local-tensor vertex-operators (9.1.8) when $D_0 = \text{flat}$, and local-tensor transformation with conformal-weight

$$h_R = \tfrac{1}{2}\mu^2(T_0(d_0), \mu(0)) + \mu(T_0(d_0), \mu(0)) \cdot D_0 \qquad (10.15)$$

is recovered in (10.13) and (10.14).

APPENDIX A: Weight Bases of g

Consider an arbitrary Hermitian representation $\{T^A_{ij}\}$ of g and its conjugate representation $\bar{T}^A_{ij} = -T^{A*}_{ij}$. The weights μ^a, $a = 1, ..., \text{rank}\, g$, and weight-basis vectors χ_μ satisfy

$$\bar{T}^a_{ij}\chi_{-\mu}(j) = -\mu^a \chi_{-\mu}(i), \qquad T^a_{ij}\bar{\chi}_\mu(j) = \mu^a \bar{\chi}_\mu(i) \qquad (A.1a)$$

$$\bar{\chi}_\mu(i)\bar{T}^a_{ij} = -\mu^a \bar{\chi}_\mu(j), \qquad \chi_{-\mu}(i) T^a_{ij} = \mu^a \chi_{-\mu}(j), \qquad (A.1b)$$

where $\bar{\chi}_\mu(i) = \chi^*_{-\mu}(i)$, and

$$\sum_i \bar{\chi}_\mu(i)\chi_{-\mu'}(i) = \delta_{\mu,\mu'}, \qquad \sum_\mu \chi_{-\mu}(i)\bar{\chi}_\mu(j) = \delta_{ij}. \qquad (A.2)$$

The left-eigenvectors transform a corresponding representation R^i of g from a Cartesian-basis (3.16b) to a weight-basis

$$R^\mu(z) = \sum_i \bar{\chi}_\mu(i) R^i(z), \qquad R^i(z) = \sum_\mu R^\mu(z)\chi_{-\mu}(i) \qquad (A.3a)$$

$$(T^a_m, R^\mu(z)) = \mu^a z^m R^\mu(z). \qquad (A.3b)$$

Real representations satisfy $\bar{T}^A_{ij} = T^A_{ij}$ and $\bar{\chi}_\mu = \chi_\mu$. The particular weight-basis $\{\chi_\alpha(i), i \neq \text{CSA}\}$ of the adjoint with $(T^A_{\text{adj}})_{BC} = -if^{ABC}$ and α a root of g provides the transformation

$$E^\alpha_m = \sum_i \chi_\alpha(i) T^i_m, \qquad T^i_m = \sum_\alpha E^\alpha_m \chi_{-\alpha}(i) \qquad (A.4)$$

from a Cartesian basis to a Cartan–Weyl basis

$$(T_m^a, T_n^b) = km\delta^{ab}\delta_{m,-n}, \qquad (T_m^a, E_n^\alpha) = \alpha^a E_{m+n}^\alpha \tag{A.5a}$$

$$(E_m^\alpha, E_n^\beta) = \begin{cases} N(\alpha, \beta) E_{m+n}^{\alpha+\beta}, & \alpha+\beta = \text{root} \\ \alpha \cdot T_{m+n} + km\delta_{m,-n}, & \alpha+\beta = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{A.5b}$$

In this case, the further relations among the eigenvectors

$$\sum_{ij} \chi_\alpha(i)(T_{\text{adj}}^k)_{ij}\chi_\beta(j) = \begin{cases} -N(\alpha, \beta)\chi_{\alpha+\beta}(k), & \alpha+\beta = \gamma \\ 0, & \text{otherwise} \end{cases} \tag{A.6a}$$

$$\sum_{\alpha+\beta=\gamma} \chi_{-\alpha}(i)\chi_{-\beta}(j)\chi_\gamma(k)N(\alpha, \beta) = -(T_{\text{adj}}^k)_{ij} \tag{A.6b}$$

$$\sum_\alpha \chi_{-\alpha}(i)\chi_\alpha(j)\alpha^a = -(T_{\text{adj}}^a)_{ij} \tag{A.6c}$$

are obtained by comparing (A.5) and (2.6) with $g_{AB} = \delta_{AB}$.

APPENDIX B: Applications of a Conjugation Identity

The conjugation identity for $\Gamma(T) \in G$ in representation T of g

$$\Gamma^{-1}(T)\hat{T}^A\Gamma(T) = \Gamma^{AB}(T_{\text{adj}})\hat{T}^B \tag{B.1}$$

holds when $(T^A, \hat{T}^B) = if^{ABC}\hat{T}^C$.

As a first application, take $\Gamma(\bar{T}) = \Omega[d, \theta, \bar{T}]$, the general twist-matrix of Section 3, and $\bar{T}^A = \hat{T}^A$. It follows that Ω commutes with the weight-matrix $h(D_0, \bar{T})$ in (3.17b),

$$\Omega^{-1}(\bar{T})h(D_0, \bar{T})\Omega(\bar{T}) = h - D_0^A\Omega^{AB}(T_{\text{adj}})\bar{T}^B = h(D_0, \bar{T}) \tag{B.2}$$

since D_0^A is an eigenvector of Ω, according to (3.5a).

As a second example, we discuss the off-CSA rotation of the arbitrarily loaded deformation $L_m[d(T_0^a), D_0(T_0^a)]$ in (2.11). A double application of (B.1) with $T^A = T_0^A$, $\hat{T}^A = T_m^A$ gives

$$\begin{aligned} L_m[d(T_0^A), D_0(T_0^A)] &\equiv \Gamma^{-1}(T_0)\, L_m[d(T_0^a), D_0(T_0^a)]\Gamma(T_0) \qquad &\text{(B.3a)} \\ &= L_m[0] + \sum_n d_{-n}^B(T_0^A)T_{m+n}^B + mD_0^B(T_0^A)T_m^B \\ &\quad + \varepsilon_m[d(T_0), D_0(T_0)] &\text{(B.3b)} \end{aligned}$$

$$\begin{aligned} \varepsilon_m[d(T_0), D_0(T_0)] = \tfrac{1}{2}k \Big\{ &\sum_n d_{-n}^B(T_0^A)d_{m+n}^B(T_0^A) + 2md_m^B(T_0^A)D_0^B(T_0^A) \\ &- D_0^B(T_0^A)D_0^B(T_0^A)\delta_{m,0} \Big\} \end{aligned} \tag{B.3c}$$

$$d_n^B(T_0^A) \equiv d_n^b(\Gamma^{aA}(T_{\mathrm{adj}})T_0^A)\Gamma^{bB}(T_{\mathrm{adj}}), \qquad D_0^B(T_0^A) \equiv D_0^b(\Gamma^{aA}(T_{\mathrm{adj}})T_0^A)\Gamma^{bB}(T_{\mathrm{adj}}) \tag{B.3d}$$

$$c(D_0(T_0^A)) = c(0) - 12kD_0^B(T_0^A)D_0^B(T_0^A). \tag{B.3e}$$

The deformations (B.3d) satisfy the constraint (2.7c) and the construction (B.3b) includes arbitrarily loaded general orbifold-ghosts (see also Section 6.2 and Appendix F).

The conjugation identity (B.1) is also employed in obtaining the local automorphism (3.13), and in Appendix D.

APPENDIX C: c-Fixed Deformation of a Bose–Fermi System

The simplest deformation begins with a single complex antiperiodic Fermi ($\tau = 1$) or Bose ($\tau = -1$) quark [2] at the origin

$$L[0] = \frac{i}{2} \,{}^{\circ}_{\circ}\bar{\psi}_{\mathrm{BH}} \overleftrightarrow{\partial}_\theta \psi_{\mathrm{BH}}{}^{\circ}_{\circ} \tag{C.1a}$$

$$\psi_{\mathrm{BH}}(z) = \sum_{n=0}^{\infty} \left(b^{\mathrm{BH}}_{n+1/2} z^{-(n+1/2)} + d^{\dagger\mathrm{BH}}_{n+1/2} z^{n+1/2}\right),$$
$$\bar{\psi}_{\mathrm{BH}}(z) = \sum_{n=0}^{\infty} \left(b^{\dagger\mathrm{BH}}_{n+1/2} z^{n+1/2} + d^{\mathrm{BH}}_{n+1/2} z^{-(n+1/2)}\right) \tag{C.1b}$$

$$(b^{\mathrm{BH}}_{m+1/2}, b^{\dagger\mathrm{BH}}_{n+1/2})_\pm = \tau(d^{\mathrm{BH}}_{m+1/2}, d^{\dagger\mathrm{BH}}_{n+1/2})_\pm = \delta_{m,n}, \qquad b^{\mathrm{BH}}_{n+1/2}\,|0\rangle_{\mathrm{BH}} = d^{\mathrm{BH}}_{n+1/2}\,|0\rangle_{\mathrm{BH}} = 0 \tag{C.1c}$$

in which the quark and antiquark are $U(1)$ representations $\bar{R} = \psi$, $R = \bar{\psi}$ with $T = -\bar{T} = 1$ of the level τ current $T(z) = {}^{\circ}_{\circ}\bar{\psi}_{\mathrm{BH}}(z)\psi_{\mathrm{BH}}(z){}^{\circ}_{\circ}$.

The explicit form of $L_0(d_0)$ in the c-fixed deformation

$$L(d_0, \theta) = L[0, \theta] + d_0 T(\theta) + \frac{\tau}{2} d_0^2 \tag{C.2a}$$

$$L_0(d_0) = \sum_{n=0}^{\infty} \left[\left(n + \frac{1}{2} + d_0\right) b^{\dagger\mathrm{BH}}_{n+1/2} b^{\mathrm{BH}}_{n+1/2} + \tau\left(n + \frac{1}{2} - d_0\right) d^{\dagger\mathrm{BH}}_{n+1/2} d^{\mathrm{BH}}_{n+1/2}\right] + \frac{\tau}{2} d_0^2 \tag{C.2b}$$

shows that all the states at the origin $\{(b^{\dagger\mathrm{BH}})^M (d^{\dagger\mathrm{BH}})^N\,|0\rangle_{\mathrm{BH}}\}$ are fixed, as discussed in the text. The modes of the boosted operators (4.1.8) are a continuous relabeling of the original modes

$$R^{\mu=-1}_{\mathrm{boost}}(d_0, z) = \psi_{\mathrm{boost}}(d_0, z) = z^{-d_0}\psi_{\mathrm{BH}}(z)$$
$$= \sum_{n=0}^{\infty} \left(b_{n+1/2+d_0} z^{-(n+1/2+d_0)} + d^\dagger_{n+1/2-d_0} z^{n+1/2-d_0}\right) \tag{C.3a}$$

$$\bar{R}^{\mu=1}_{\text{boost}}(d_0, z) = \bar{\psi}_{\text{boost}}(d_0, z) = z^{d_0}\bar{\psi}_{\text{BH}}(z)$$

$$= \sum_{n=0}^{\infty} (b^\dagger_{n+1/2+d_0} z^{n+1/2+d_0} + d_{n+1/2-d_0} z^{-(n+1/2-d_0)}) \tag{C.3b}$$

$$b_{n+1/2+d_0} = b^{\text{BH}}_{n+1/2}, \quad b^\dagger_{n+1/2+d_0} = b^{\dagger\text{BH}}_{n+1/2}, \quad d_{n+1/2-d_0} = d^{\text{BH}}_{n+1/2}, \quad d^\dagger_{n+1/2-d_0} = d^{\dagger\text{BH}}_{n+1/2} \tag{C.3c}$$

so the states of each deformed module are a continuous relabeling of the module at the origin. Since $T_{\text{boost}}(d_0, z) = T(z) + \tau d_0$ does not twist, the identity of each highest-weight-state is maintained throughout the deformation. The ground-state in the Fermi case

$$\prod_{n=0}^{m} d^{\dagger\text{BH}}_{n+1/2} |0\rangle_{\text{BH}} = \prod_{n=0}^{m} d^\dagger_{n+1/2-d_0} |0\rangle_{\text{BH}}, \qquad m+\tfrac{1}{2} < d_0 < m+\tfrac{3}{2} \tag{C.4a}$$

$$|0\rangle_{\text{BH}}, \qquad |d_0| < \tfrac{1}{2} \tag{C.4b}$$

$$\prod_{n=0}^{m} b^{\dagger\text{BH}}_{n+1/2} |0\rangle_{\text{BH}} = \prod_{n=0}^{m} b^\dagger_{n+1/2+d_0} |0\rangle_{\text{BH}}, \qquad -(m+\tfrac{1}{2}) > d_0 > -(m+\tfrac{3}{2}) \tag{C.4c}$$

is passed from one highest-weight state to another at the degeneracy points $d_0 = \mathbb{Z} + \frac{1}{2}$, which are two-fold degenerate complex Ramond [4] vacua. The highest-weight states in the Bose case

$$(b^{\dagger\text{BH}}_{1/2})^M |0\rangle_{\text{BH}}, h = M(d_0 + \tfrac{1}{2}); \qquad (d^{\dagger\text{BH}}_{1/2})^M |0\rangle_{\text{BH}}, h = -M(d_0 - \tfrac{1}{2}) \tag{C.5}$$

are bottomless for $|d_0| > \frac{1}{2}$.

Another form of the deformation (C.2) is obtained with (C.3) in terms of the boosted quarks

$$L(d_0) = \tfrac{i}{2} \,{}^\circ_\circ \bar{\psi}_{\text{boost}}(d_0) \overleftrightarrow{\partial}_\theta \psi_{\text{boost}}(d_0) {}^\circ_\circ{}_{\text{re}} + \varepsilon_{\text{re}}(d_0) \tag{C.6a}$$

$$\varepsilon(d_0)_{\text{re}} = \frac{\tau}{2} \left[d_0 - \text{int}\left(d_0 + \frac{1}{2}\right)\right]^2 \tag{C.6b}$$

$$L_0(d_0) = \sum_{n=0}^{\infty} \left[\left(n + \frac{1}{2} + d_0\right) b^\dagger_{n+1/2+d_0} b_{n+1/2+d_0} + \tau\left(n + \frac{1}{2} - d_0\right) d^\dagger_{n+1/2-d_0} d_{n+1/2-d_0}\right] + \frac{\tau}{2} d_0^2, \qquad |d_0| < \frac{1}{2} \tag{C.6c}$$

$$= \sum_{n=-\text{int}(d_0+1/2)}^{\infty} \left(n + \frac{1}{2} + d_0\right) b^\dagger_{n+1/2+d_0} b_{n+1/2+d_0}$$

$$-\tau \sum_{0 \leq n < -\mathrm{int}(d_0+1/2)} \left(n+\frac{1}{2}+d_0\right) b_{n+1/2+d_0} b^{\dagger}_{n+1/2+d_0}$$

$$+\tau \sum_{n=0}^{\infty} \left(n+\frac{1}{2}-d_0\right) d^{\dagger}_{n+1/2-d_0} d_{n+1/2-d_0} + \varepsilon_{\mathrm{re}}(d_0), \qquad d_0<0 \quad \text{(C.6d)}$$

$$T_{\mathrm{boost}}(d_0) = :\bar{\psi}_{\mathrm{boost}}(d_0)\psi_{\mathrm{boost}}(d_0):_{\mathrm{re}} + \tau[d_0+\mathrm{int}(-d_0+\tfrac{1}{2})-\tfrac{1}{2}\delta_{d_0,\mathbb{Z}+1/2}], \quad \text{(C.6e)}$$

where the re-normal-ordering of the boosted operator terms (anti)symmetrizes with respect to (Fermi) Bose zero-modes. The relations (C.6a), (C.6b), (C.6d), (C.6e) show that the system has unit period in d_0, and (C.6c) was given for the fermions in [56].

APPENDIX D: Rotation of Flat-Deformed Systems

We discuss Cartesian frame-rotation for the general flat deformation $L_m[[d, D_0; T]$ in (2.7) of a Sugawara construction for arbitrary level of g at the origin. The conjugation identity of Appendix B is freely used. Define rotated currents and deformations

$$T^A_m(\Gamma) \equiv \Gamma(T_0) T^A_m \Gamma^{-1}(T_0) = T^B_m \Gamma^{BA}(T_{\mathrm{adj}}) \quad \text{(D.1a)}$$

$$d^A_m(\Gamma) \equiv d^B_m \Gamma^{BA}(T_{\mathrm{adj}}), \qquad D^A_0(\Gamma) \equiv D^B_0 \Gamma^{BA}(T_{\mathrm{adj}}) \quad \text{(D.1b)}$$

for $\Gamma(T_0) \in G$. Then equality of the deformed Virasoro operators in the two frames

$$L_m[d, D_0; T] = L_m[d(\Gamma), D_0(\Gamma); T(\Gamma)] \quad \text{(D.2)}$$

follows since $\Gamma(T_{\mathrm{adj}})$ is orthogonal and the Sugawara construction is invariant under $SO(\dim g)$. As an application, Eqs. (D.1b) and (D.2) may be used with the appropriate Γ to rotate any particular deformation mode, say d^A_0, onto the CSA.

The boosted currents (3.6a) of $L_m[d, D_0; T]$ and $L_m[d(\Gamma), D_0(\Gamma); T(\Gamma)]$ are

$$T^A_{\mathrm{boost}}[d, \theta] = \Omega^{AB}[d, \theta, T_{\mathrm{adj}}](T^B(\theta) + k d^B(\theta)) \quad \text{(D.3a)}$$

$$T^A_{\mathrm{boost}}[\Gamma; d(\Gamma), \theta] = \Omega^{AB}[d(\Gamma), \theta, T_{\mathrm{adj}}](T^B(\Gamma; \theta) + k d^B(\Gamma; \theta)), \quad \text{(D.3b)}$$

respectively, while the conjugation identity

$$d(\Gamma; \theta) \cdot T_{\mathrm{adj}} = \Gamma^{-1}(T_{\mathrm{adj}}) d(\theta) \cdot \mathrm{T}_{\mathrm{adj}} \Gamma(T_{\mathrm{adj}}) \quad \text{(D.4)}$$

and the definition (3.4) relate the twist-matrices in the two frames according to $\Omega[d(\Gamma)] = \Gamma^{-1}\Omega[d]\Gamma$. Then the relation between the boosted currents of the two frames

$$T^A_{\mathrm{boost}}[\Gamma; d(\Gamma), \theta] = T^B_{\mathrm{boost}}[d, \theta] \Gamma^{BA}(T_{\mathrm{adj}}) \quad \text{(D.5)}$$

follows with (D.3). The rotation identity (D.5) is applied in Section 4.2.

APPENDIX E: MAGNETIC-ANALOGUE PICTURE AND $(\tilde{d}_0)_{\text{eff}}$

The simplest magnetic-analogue twist is Gottfried's [61] soluble model for $g = SU(2)$ with $d^A = (d^3, d^1, d^2) = (\omega_0, B_1 \cos\theta, -B_1 \sin\theta)/\sqrt{2}$ in which we have computed

$$\sigma_\mu(d) = \sigma_0 - \mu(1+\Delta)/\sqrt{2}, \qquad \Delta = \sqrt{(1-\omega_0)^2 + B_1^2} \tag{E.1}$$

for the moding $\sigma/\mathbb{Z}$ of the homogeneous boosted operators (3.10), (3.20b) corresponding to weight μ of Hermitian representation T. The Cartesian frame is homogeneous on resonance ($\omega_0 = 1$) with $B_1 = \frac{1}{2}$, and the $SU(2)$-orbifold automorphism (F.4) is verified for the boosted currents in this case.

A homogeneous frame is obtained in the general case by rotating $\Omega(2\pi)$ onto the CSA and using the weight-basis of Appendix A. The generic moding is

$$\sigma_\mu[d] = \sigma_0 - \mu \cdot (\tilde{d}_0)_{\text{eff}}[d] \tag{E.2}$$

in terms of an equivalent effective CSA zero-mode deformation $(\tilde{d}_0)_{\text{eff}}$, which is difficult to obtain in closed form.

For small deformations, the result

$$\begin{aligned}(\tilde{d}_0^a)_{\text{eff}} = \tilde{d}_0^a &- \frac{i}{2} f^{aBC} \sum_{m \neq 0} \frac{1}{m} d^B_{-m} d^C_m + \frac{1}{2} f^{aBC} f^{CDe} \sum_{m \neq 0} \frac{1}{m^2} d^B_{-m} d^D_m \tilde{d}^e_0 \\ &+ f^{aBC} f^{CDE} \sum_{\substack{m>0, n>0 \\ m \neq n}} \frac{1}{mn} d^B_{-m} d^D_n d^E_{m-n} + O(d^4)\end{aligned} \tag{E.3}$$

is obtained from $\Omega(2\pi)$, and the same result is obtained by comparison of the effective highest-weight shift $h((\tilde{d}_0)_{\text{eff}}) = h[0] + (\tilde{d}_0)_{\text{eff}} \cdot p + \frac{1}{2}k(\tilde{d}_0)^2_{\text{eff}}$ with a computation of the actual highest-weight shift by the method of Dalgarno and Lewis [62]. The coincidence leaves little doubt that general c-fixed flat deformations are equivalent[23] to the other pictures, but we have not found a comprehensive proof.

We also remark that the moding $\sigma_\mu[d]$ involves the holonomy of the space of deformations, since each particular deformation $d^A(\theta)$ defines a closed path C in the space, as θ moves through a period The holonomy of magnetic systems is often studied in the adiabatic approximation, when $|\tilde{d}_0| \ll N$ with N the largest mode-number of the deformation. Then

$$\sigma_\mu[d] = \sigma_0 - \frac{1}{2\pi}\left(\int_0^{2\pi} d\theta\, E_\mu(\theta) + \gamma_\mu[C]\right), \qquad \gamma_\mu[C] = \oint_C d(d^A) \langle \mu, d|\, i\frac{\partial}{\partial d^A} |\mu, d\rangle \tag{E.4}$$

[23] The Dalgarno–Lewis operator $\Lambda_1 = \sum_{m \neq 0} d^A_{-m} T^A_m / m$ is the first term $\Lambda = \Lambda_1 + \cdots$ of a unitary equivalence-transformation $\exp \Lambda$ (from the general case to $(\tilde{d}_0)_{\text{eff}}$) which we have constructed explicitly thru $O(d^2)$.

is obtained, where $E_\mu(\theta)=\mu\cdot\omega(\theta)$ is the instantaneous eigenvalue of H corresponding to the instantaneous eigenstate $|\mu, d\rangle$ and $\gamma_\mu[C]$ is Berry's phase [63], currently known only for $g=SU(2)$.

APPENDIX F: AN ORBIFOLD-GHOST SYSTEM

The simplest orbifold-ghost system is an involution of $SU(2)$,

$$L_m(\hat{d}_0, \hat{D}_0)=L_m[0]+\left(\frac{1}{2\sqrt{2}}+m\hat{D}_0\right)T^1_m+\frac{1}{2}k\left(\frac{1}{8}-\hat{D}_0^2\right)\delta_{m,0} \tag{F.1a}$$

$$\hat{d}_0^A=(\hat{d}_0^3, \hat{d}_0^1, \hat{d}_0^2)=(0,1,0)/2\sqrt{2}, \qquad \hat{D}_0^A=(0,\hat{D}_0,0) \tag{F.1b}$$

$$c(\hat{D}_0)=3k\left(\frac{1}{k+2}-4\hat{D}_0^2\right), \tag{F.1c}$$

with $L_m[0]$ the Sugawara construction for arbitrary level of $SU(2)$ at the origin. The orbifold currents may be considered with (F.1) and (3.6a) as a boost from the origin, or with (4.2.1) as a view of the torus-ghost

$$\tilde{d}_0^A=(1,0,0)/2\sqrt{2}, \qquad \tilde{D}_0^A=(\hat{D}_0,0,0) \tag{F.2}$$

from the orbifold frame. The explicit form of the rotation (4.2.1) is

$$\Gamma(T_{\text{adj}})=\exp\left(-i\frac{\pi}{2\sqrt{2}}T^2_{\text{adj}}\right)=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{F.3a}$$

$$\begin{pmatrix}T^3_{\text{boost}}(\hat{d}_0)_{m+1/2}\\ T^1_{\text{boost}}(\hat{d}_0)_m\\ T^2_{\text{boost}}(\hat{d}_0)_{m+1/2}\end{pmatrix}=\begin{pmatrix}0 & -1/\sqrt{2} & -1/\sqrt{2}\\ 1 & 0 & 0\\ 0 & -i/\sqrt{2} & i/\sqrt{2}\end{pmatrix}\begin{pmatrix}T^{3,\text{torus}}_{\text{boost}}(\tilde{d}_0)_m\\ E^{+\sqrt{2},\text{torus}}_{\text{boost}}(\tilde{d}_0)_{m+1/2}\\ E^{-\sqrt{2},\text{torus}}_{\text{boost}}(\tilde{d}_0)_{m+1/2}\end{pmatrix} \tag{F.3b}$$

and the resulting orbifold automorphism

$$\begin{aligned}&(T^A_{\text{boost}}(\hat{d}_0)_{m+\sigma_A}, T^B_{\text{boost}}(\hat{d}_0)_{n+\sigma_B})\\ &\quad=i\sqrt{2}\,\varepsilon^{ABC}T^C_{\text{boost}}(\hat{d}_0)_{m+n+\sigma_A+\sigma_B}+k(m+\sigma_A)\delta^{AB}\delta_{m+\sigma_A,-n-\sigma_B}\end{aligned} \tag{F.4}$$

with $\sigma_1=0$, $\sigma_2=\sigma_3=\frac{1}{2}$ is Cartesian in this case.

The charged currents of the torus-ghost have weights $h(\pm\sqrt{2}, \tilde{D}_0)=1\pm\sqrt{2}\,\hat{D}_0$ and $T^{3,\text{torus}}_{\text{boost}}$ is not a tensor. It follows in particular that the antiperiodic Cartan current [34, 35] $T^3_{\text{boost}}(\hat{d}_0)=-(E^{+\sqrt{2},\text{torus}}_{\text{boost}}(\tilde{d}_0)+E^{-\sqrt{2},\text{torus}}_{\text{boost}}(\tilde{d}_0))/\sqrt{2}$ of the orbifold-ghost has indefinite conformal-weight, as noted in the text.

The equivalent form of the deformation

$$L_m(\hat{d}_0, \hat{D}_0) = \tfrac{1}{2} {}^\times_\times (T^3_{\rm boost}(\hat{d}_0))^2 {}^\times_\times{}_m + m\hat{D}_0 T^1_{\rm boost}(\hat{d}_0)_m + \tfrac{1}{2}(\tfrac{1}{8} - \hat{D}_0^2)\delta_{m,0} \quad \text{(F.5)}$$

follows for level-one because the difference of (F.5) and (F.1a) is a Virasoro operator with $c=0$ [18]. With the further assumption of antiperiodic free fermions and deformation currents $T^A = \bar{\psi}\tau^A\psi/\sqrt{2}$ at the origin, Eqs. (4.1.8), (4.1.9), and (F.3b) express the orbifold currents

$$T^A_{\rm boost}(\hat{d}_0) = (\varepsilon^{A2B} + \delta^{A2}\delta^{B2}) {}^\circ_\circ \bar{\psi}^{\rm torus}_{\rm boost}(\tilde{d}_0) \frac{\tau^B}{\sqrt{2}} \psi^{\rm torus}_{\rm boost}(\tilde{d}_0) {}^\circ_\circ{}_{\rm re} + \frac{\delta^{A1}}{2\sqrt{2}} \quad \text{(F.6)}$$

in terms of the $\frac{1}{4}$-integral-moded fermions $\psi^{\rm torus}_{\rm boost}(\tilde{d}_0)$ of the torus-ghost.

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Note added in proof. We have learned that a $SL(2, \mathbb{R})$ example of the flat c-changing deformations was given by A. A. Belavin in his unpublished talk at the 1987 Kyoto Superstring Workshop. Some months after our preprint, two other papers on deformations appeared: The flat deformations have been independently rediscovered by N. Sakai and P. Suranyi (*Imperial I.T.P.* **87–88** (1988), 36) and deformations are discussed in A. A. Beilinson and V. V. Schechtman "Determinant Bundles and Virasoro Algebras," Moscow, 1988.

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