Incorporating the Stern-Gerlach delayed-choice quantum eraser into the undergraduate quantum mechanics curriculum

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Incorporating the Stern-Gerlach delayed-choice quantum eraser into the undergraduate quantum mechanics curriculum

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As "Stern-Gerlach first" becomes increasingly popular in the undergraduate quantum mechanics curriculum, we show how one can extend the treatment found in conventional textbooks to cover some exciting new quantum phenomena. Namely, we illustrate how one can describe a delayed choice variant of the quantum eraser which is realized within the Stern-Gerlach framework. Covering this material allows the instructor to reinforce notions of changes in basis functions, quantum superpositions, quantum measurements, and the complementarity principle as expressed in whether we know "which-way" information or not. It also allows the instructor to dispel common misconceptions of when a measurement occurs and when a system is in a superposition of states.
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I. INTRODUCTION

The Stern-Gerlach experiment was originally performed[1] by Otto Stern and Walter Gerlach in 1922. While it can be thought of as simply an experiment to separate an atomic beam into its different projections of angular momentum, the Stern-Gerlach experiment also illustrates a number of different quantum phenomena. One can use it to show that quantum mechanics requires a probabilistic interpretation. One can use it to show that quantum states cannot simultaneously have definite projections of angular momentum on two non-collinear axes. It also acts as one of the simplest paradigms of a two-state quantum system (for the case of a spin-one-half atom like silver), illustrating the discreteness of quantum eigenvalues.

Educators have long realized the importance of this experiment.[2–7] It has appeared in many textbooks. Here, we highlight a few texts that bring this experiment to the forefront, by employing it as one of the first quantum experiments that a student encounters. These texts deviate from the far more common norm of covering quantum mechanics from a historical perspective[8,9] or by starting with the wave equation in coordinate space.[10] We believe that there are significant advantages to proceeding in this Stern-Gerlach first methodology, as it allows the students to encounter experiments that they can easily analyze right from the beginning. Furthermore, as we show here, one can extend those treatments to allow the students to encounter sophisticated quantum paradoxes even before they learn what a coordinate-space wavefunction is.

The Feynman Lectures on Physics[2] introduces the Stern-Gerlach experiment quite early in its discussion of quantum mechanics, actually covering the spin-one case before the spin-one-half case. This text also describes what we will call the Stern-Gerlach analyzer loop (following Styer,[4] see below); this device is sometimes called a Stern-Gerlach quantum eraser by other authors,[5] but we will be reserving that language for the more complex eraser we describe below. It is at this stage that most educators (including us) move from the real Stern-Gerlach experiment to more complicated "experiments" that invoke the principles of the Stern-Gerlach experiment, but do so in a more complex format than can actually be carried out in a laboratory; we will use quotes to describe experiments that can in principle be carried out, but to our knowledge have never actually been performed in a laboratory. Sakurai employed the Stern-Gerlach experiment early in his textbook[3] and used it to also discuss the Bell experiments. Our treatment of the subject is influenced most by Styer's wonderful text *The Strange World of Quantum Mechanics*,[4] which introduces a number of complex quantum ideas with the Stern-Gerlach experiment. These are the two-slit experiment, Wheeler's delayed choice, the Einstein-Podolsky-Rosen paradox, and the Bell experiments. Styer's text also carefully describes the classical version of the experiment, which is critical for students to master in order to appreciate the quantum nature of the real experiment.

Three recent undergraduate textbooks, Townsend's *A Modern Approach to Quantum Mechanics*,[5] McIntyre, Manogue and Tate's *Quantum Mechanics: A Paradigms Approach*,[6] and Beck's *Quantum Mechanics: Theory and Experiment*,[7] all adopt the Stern-Gerlach first paradigm, introducing students to this experiment as their initial (or early) encounter with quantum mechanics. While these texts move on to a more conventional style of quantum treatment afterwards, this critical change allows students to dive into a quantum system that they can understand all aspects of and allows them to lean on this knowledge as they learn about new and different quantum phenomena in the remainders of the books.

This article is organized as follows: (i) in Sec. II, we provide a short history of the Stern-Gerlach experiment, delayed choice, the quantum eraser, and their use in quantum mechanics pedagogy; (ii) in Sec. III, we describe the different apparatus needed for the Stern-Gerlach experiments and how one employs them in instruction; (iii) in Sec. IV, we describe the details of how to create and analyze a delayed choice Stern-Gerlach quantum eraser; (iv) in Sec. V, we discuss possible experimental implementations; and (v) in Sec. VI, we present our conclusions.

II. BRIEF HISTORY OF STERN-GERLACH EXPERIMENT PEDAGOGY

Quantum mechanics has seen numerous developments that have not yet made it into most introductory quantum texts. For example, in the 1980s, John Wheeler introduced the notion of delayed choice,[11,12] where an experimental apparatus is modified *while the particle is moving through it*, in such a way that the modification post-selects what type of measurement will be performed. Wheeler hypothesized that these types of experiments, which can determine whether a particle goes through just one slit, or two slits at the same time, in a two-slit experiment, have the spooky behavior of acting like the quantum particle is able to influence what has already occurred, by going backwards in time. It turns out that this awkward notion is easily dispelled when one properly interprets when the system is in a superposition of states and precisely when a measurement collapses the wavefunction.[13] Nevertheless, the notion of a delayed choice experiment being employed to change the outcome is a remarkably powerful demonstration, as can be seen by numerous videos available on the internet, which illustrate this phenomenon using crossed polarizers over each slit of the two-slit experiment and an additional polarizer, whose orientation can be rotated, just before the light hits the detector screen.[14] Those videos are actually showing a delayed-choice quantum-eraser variant.

The quantum eraser idea of Scully *et al.*[15,16] is even more fascinating. Here, what is generally done is that the particles that are input into a two-slit experiment (or a Mach-Zehnder interferometer) are also entangled with other quantum particles, which can be employed to provide which-way information about how the particle moves through the device. As long as the entanglement with the other particle persists, the conventional interference effects are suppressed. But if the entanglement with the other particle is removed, then the interference effects also return. What is remarkable about these experiments is that they often can have the choice for whether we see the interference or not decided well after the quantum particles have gone through the device. One can think of the delayed-choice aspect as providing a filter that removes the results of the experiment which do not provide the interference one is trying to "restore." The interference is then never fully restored because the entanglement and subsequent filtering always remove some particles from the experiment, and so the interference oscillations have a smaller amplitude than what one would see if there was never any entanglement in the first place.

If the Stern-Gerlach-first trend continues, an increasing number of students will be exposed to the Stern-Gerlach experiment early in the quantum curriculum. It is for this reason that we show how one can employ these experiments to cover quite advanced and fascinating phenomena, early on in a course. This then allows students to experience the truly strange behavior that lies within quantum mechanics and to know that it can be quantitatively described within the theory.

We end this section with a brief discussion of the history of pedagogy of the Stern-Gerlach experiment in particular and quantum erasers in general. The Stern-Gerlach experiment first entered pedagogy with Wigner's classic article where the analyzer loop was introduced in 1963.[17] Scully *et al.* performed an in-depth analysis of the analyzer loop to show that it should be thought of as creating a superposition of states unless a measurement is performed on it to determine which-way information.[18] In addition, a series of papers have provided detailed calculations of the dynamics of the Stern-Gerlach experiment, paying particular attention to the fact that the magnetic field must have a component perpendicular to the direction where the atomic beam is split due to the fact that the field is divergenceless.[19–24] A tutorial has also been created to directly confront common misconceptions about the experiment.[25] Finally, an example of a quantum eraser, using quite different methodology from what we propose here (crossed Stern-Gerlach analyzers with a two-slit experiment in between), has also appeared.[26]

The quantum eraser has been discussed within many different platforms. The simplest demonstrations use polarization of light within a two-slit experiment[27,28] (including an experimental setup[29]). Similarly, a Mach-Zehnder interferometer[30] can be used to also illustrate the quantum eraser. Previous work includes a tutorial[31] and descriptions of undergraduate experimental apparatus without[32,33] and with[34] a delayed choice option added. While all quantum erasers share some form of similarity with each other in terms of how the which-way information is tagged, the details for how these different devices work and for the different methodologies employed for pedagogy separate the different discussions. We complete the cycle with this work by providing a delayed choice quantum-eraser discussion within the Stern-Gerlach framework.

III. PRELIMINARIES FOR THE STERN-GERLACH EXPERIMENT

The idea for an accessible Stern-Gerlach quantum eraser began when we introduced the concept into a massive open on-line course (MOOC) entitled *Quantum Mechanics For Everyone* which ran on edX from April 2017 until March 2019.[35] The MOOC was intended for all audiences and so did not employ the full abstract quantum formalism. Freericks was the lead instructor and course developer, Vieira created over half of the computer-based tutorials that run under JavaScript,[36] and Courtney was in the original student cohort. Since this MOOC will remain available as an archived resource on edX, we describe how it covers the Stern-Gerlach experiment to define the terminology and to introduce the different devices we need to describe the delayed-choice quantum-eraser variant. As mentioned above, this treatment is heavily influenced by both Styer's[4] and Feynman's[2] approaches. The experience with the MOOC showed us that similar ideas could be presented more broadly within the undergraduate curriculum and this is our emphasis here.

To begin, students need to understand how a classical Stern-Gerlach experiment works, which involves shooting a beam of current loops through an inhomogeneous magnetic field. Using the right hand rule and curling ones' fingers in the orientation of current flow through the loop, the thumb points toward the north pole of an effective magnet that represents the current loop. As Styer shows,[4] one can next develop that a current loop precesses in a magnetic field, with a constant projection of the effective magnet onto the field axis, and it feels a force if the magnetic field is inhomogeneous in space. It is important that the students recognize that one needs an inhomogeneous field to apply a force proportional to the projection and that the projection does not change during the time the current loop is in the field. This then means that the net deflection of the current loop upwards or downwards varies monotonically with the projection that the effective magnet makes with the magnetic field.

Hence, a classical beam of current loops shot through an inhomogeneous field will fan out according to the different projections of the effective magnet onto the field axis, with the spatial position correlated with the magnitude of the projection. One can describe such an experiment as analogous to a triangular prism, which separates white light according to its color. An example of such a classical Stern-Gerlach experiment is shown in Fig. 1, where the fan-out path for one projection of the effective magnet is plotted.

Of course, the quantum experiment does not produce a continuous beam of separated projections. When run using silver atoms, it shows just two different projections of the angular momentum: one corresponding to $+1/2\,\mu_B$ and the other to $-1/2\,\mu_B$, with μ_B being the Bohr magneton. This quantum result motivates a number of follow-up experiments to understand this phenomenon. We begin by showing how one packages the Stern-Gerlach analyzer for use in further experiments (see Fig. 2). Since the quantum Stern-Gerlach experiment on silver produces only two results, regardless of the orientation of the analyzer, we think of the experiment as a separation region where the magnets are positioned and "tubes" that collect the atoms according to their projections and direct them to the respective + and − exits (curving their velocities to be horizontal). The device is packaged together so that we have a direction of the field given by the arrow, the sense of the inhomogeneity of the field also given by the widening of the arrow's shaft, and the two exits (one with a positive projection on the axis, labeled +, and the other with a negative projection on the axis, labeled −). The tubes that curve to the exits can be thought of as being constructed from an inhomogeneous magnet oriented opposite to the initial separating magnet, which curves the paths to be horizontal and ejects the atomic beams in a horizontal direction after emerging from the analyzer.

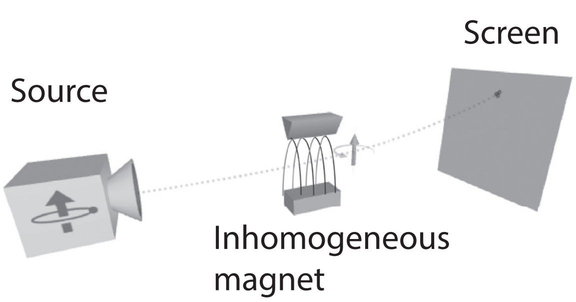

Fig. 1. Schematic of the classical Stern-Gerlach experiment, with an unpolarized source of classical current loops, an inhomogeneous magnetic field generated between the shaped magnetic poles with field lines sketched, and a screen to detect the projection of the current loops as they move through the device. The curved dashed line indicates one possible current loop trajectory. This current loop has a maximal projection on the z-axis, and so it is deflected the furthest upwards.

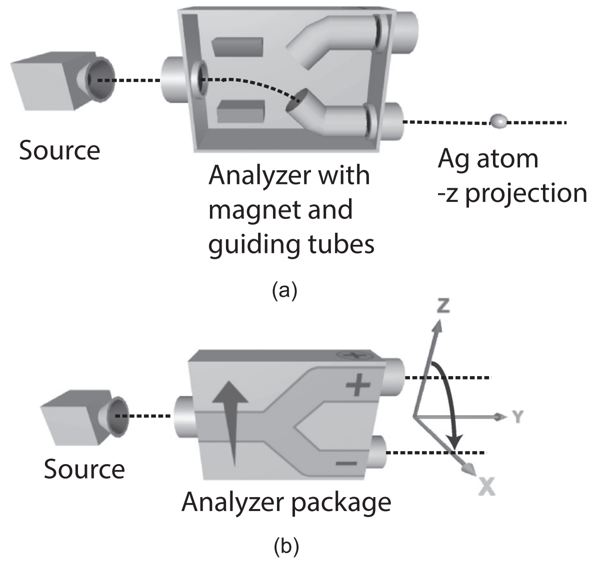

Fig. 2. Schematic of the quantum Stern-Gerlach experiment with silver atoms, which produces only two deflections. Both images start with an unpolarized source of silver atoms on the top. The packaging with an inhomogeneous magnet and "guiding tubes" on the top image is covered with a schematic annotation creating the Stern-Gerlach analyzer in the right image, which also illustrates the coordinate system used to describe the orientation of the analyzer (here, the orientation corresponds to an angle in the x–z plane as indicated by the curved arrow). The dotted lines indicate the path an atom with a negative projection takes through the device (top) and schematically shows the two possible paths that an initially unpolarized atom can take (bottom).

The Stern-Gerlach analyzer can then be employed in a series of experiments (see Fig. 3); as far as we know, none of these experiments have ever been performed in a lab. In these series of three experiments (all starting from an unpolarized atom source), we measure results following one specific path of atoms through the device, determined by matching the output of one analyzer into the input of another. The dotted lines show these paths explicitly. The axis orientation of each analyzer is denoted by the direction oriented as an angle in the x-z plane of the increasing magnetic field (or, equivalently, the direction that the positive-projection exit is oriented in). We use an overbar to denote an analyzer oriented along the corresponding negative axis—hence, x denotes an analyzer with a magnetic field in the positive x direction (horizontal and out of the page), while $\bar{z}$ denotes an analyzer oriented along the negative z direction (vertical and downward).

In the first "experiment" (see Fig. 3), we measure on z and on z again (top panel), by taking the beam of atoms emerging from the negative exit (negative projection on the z-axis) and measuring their projection again (finding it retains a negative projection on the z-axis); a similar experiment can be done with atoms that have a positive projection. The results are told to the students, and this shows that the Stern-Gerlach analyzer measurements are reproducible, in the sense that an atom with a definite projection continues to have the same definite projection. In the second experiment (see Fig. 3), we measure on z and then on $\bar{z}$, to see the relationship between measuring on axes oriented opposite to each other (center panel). The results of this experiment are also told to the students, and this shows that all atoms with a negative projection on the z-axis will have a positive projection on the $\bar{z}$-axis and vice versa because all atoms exit the opposite exit

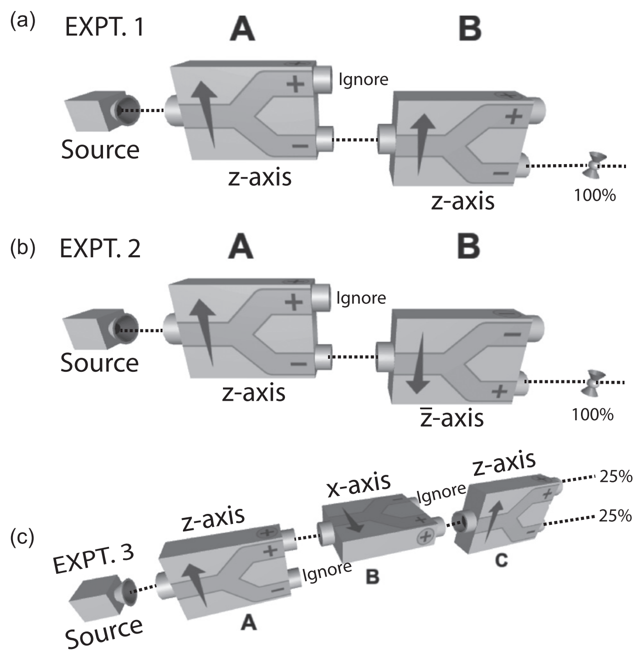

Fig. 3. Three different experiments with Stern-Gerlach analyzers all starting from an unpolarized source. (Top) experiment 1, measure on the z-axis (a), capture all from the $-$ exit, and then measure on the z-axis again (b). This experiment shows that measurements on an analyzer are reproducible. (Middle) experiment 2, measure on the z-axis (a), capture all from the $-$ exit, and then measure on the $\bar{z}$-axis (b). This experiment shows that a positive projection on one axis is a negative projection on the inverse axis and vice versa. (Bottom) experiment 3, measure on the z-axis (a), capture all from the $+$ exit, measure on the x-axis (b), capture all from the $+$ exit, and then measure on the z-axis again (c). Since we only input atoms with a positive z-axis projection into (b), one might expect that they will all emerge with a positive projection on the z-axis from (c), but we find they are equally likely to emerge with a positive or negative projection on the z-axis, because the angular momentum operators in different Cartesian directions are incompatible operators, and the atom can have a definite projection on only one axis at any given moment.

on the second analyzer. Hence, knowing the projection on one axis means we also know the projection on the axis that is flipped by 180°. In the third experiment, we measure on z, then on x, and then on z again. We must also tell the students the results of this experiment, which is that the atoms emerge with equal amounts in the up exit and in the down exit. This experiment shows that atoms can only have a projection on the last axis on which they were measured (right panel). In other words, if the atoms always enter the horizontal analyzer (B) with a positive projection on the z-axis, then they exit (B) with no definite projection on the z-axis anymore, for we find they can emerge from the final analyzer (C) either from the $+$ or $-$ exit of the vertical analyzer. Since we cannot predict with certainty which exit each will emerge from, we are forced into a probabilistic interpretation of these quantum experiments. We cannot foretell the outcome of any single experimental trial—theoretically, we only know the probability for exiting each exit, while experimentally, we require many trials to amass enough data to estimate those probabilities. The third experiment also illustrates the principle that incompatible operators cannot have simultaneous eigenvalues—as we learn that we cannot have a state with a definite z-axis projection *and* a definite x-axis projection—the atom has a definite projection on only one axis (the last one it was measured on).

We illustrate now how one can perform a detailed analysis of these experiments. We begin by employing Dirac bra-ket notation, where a bra $\langle\psi|$ and a ket $|\psi\rangle$ are the notations for a quantum state ψ. Forming a bra-ket, such as $\langle\psi'|\psi\rangle$, corresponds to the inner product between the two different states. One can simply think of the bra and the ket as being place holders for the labels that denote the different states.

In order to analyze the experiments, we need three postulates: (i) the norm of all quantum states is 1, and so $\langle\psi|\psi\rangle = 1$; (ii) the measurement by a Stern-Gerlach analyzer corresponds to a projection of $|\psi\rangle$ onto the state corresponding to the exit of the analyzer (for example, $|\uparrow;z\rangle\,\langle z;\uparrow|$ is the projector onto the positive projection atomic state along the z-axis with the up arrow denoting a positive projection and the down arrow a negative projection, as is common with spin-one-half systems); and (iii) the modulus squared of the final projected wavefunction yields the probability to emerge from a corresponding exit of a Stern-Gerlach analyzer. Note that all quantum states are unit norm, but a projected wavefunction corresponds to a quantum state multiplied by a scalar whose magnitude is less than or equal to one. We assume that other standard results can be developed as needed (eigenstates with different eigenvalues are orthogonal, how to express eigenstates of different Pauli matrices, etc.) and do not discuss them further here.

Using this formalism, we have for experiment 1 (Fig. 3, top) the following analysis. We think of the unpolarized source plus Stern-Gerlach analyzer (A) as a polarized source of atoms. Then, the initial state entering analyzer (B) is a down spin state $|\downarrow;z\rangle$. After passing through the analyzer (B), we perform the standard measurement procedure. The probability to exit the $+$ exit of the z-oriented analyzer is the norm squared of the appropriately projected state $|\uparrow;z\rangle\langle z;\uparrow|\downarrow;z\rangle$ (which is zero). and the probability to exit the $-$ exit of the z-oriented analyzer is the norm squared of the corresponding projected state $|\downarrow;z\rangle\,\langle z;\downarrow|\downarrow;z\rangle = |\downarrow;z\rangle$ (which is one). Hence, all atoms that enter the analyzer (B) exit the $-$exit.

Using the identities that $|\uparrow;z\rangle = |\downarrow;\bar{z}\rangle$ and $|\downarrow;z\rangle = |\uparrow;\bar{z}\rangle$ allows us to analyze experiment 2 (Fig. 3, middle). The wavefunction after emerging through the first analyzer is in $|\downarrow;z\rangle$ because we examine only the atoms exiting the $-$ exit of (A). Then, we find we need to evaluate $|\uparrow;\bar{z}\rangle\,\langle\bar{z};\uparrow|\downarrow;z\rangle$ as the projected state for exiting the $+$ exit of the $\bar{z}$ oriented analyzer. Replacing the states labeled on the $\bar{z}$ axis, by their z-axis counterparts, yields $|\downarrow;z\rangle\,\langle z;\downarrow|\downarrow;z\rangle = |\downarrow;z\rangle$. Squaring gives a probability of 1, and hence, all atoms that enter the analyzer (B) exit its $+$ exit, which can be directly confirmed by calculating the projected state and probability to emerge from the $-$ exit of (B).

For the last experiment (Fig. 3, bottom), we need to know the representation of the x-states in terms of the z-states: $|\uparrow;x\rangle = 1/\sqrt{2}(|\uparrow;z\rangle + |\downarrow;z\rangle)$, which can be easily developed through the spin operators and their properties. Then, we have that the wavefunction of the system after exiting the first analyzer (A) is $|\uparrow;z\rangle$; we compute all probabilities below relative to the atoms entering the analyzer (B). The wavefunction exiting the $+$ exit of the x-axis analyzer (B) is then $|\uparrow;x\rangle\,\langle x;\uparrow|\uparrow;z\rangle = 1/2\,(|\uparrow;z\rangle + |\downarrow;z\rangle)(\langle z;\uparrow| + \langle z;\downarrow|)\,|\uparrow;z\rangle$; the projection postulate is used because the analyzer always performs a measurement. Using the fact that $\langle z;\uparrow|\downarrow;z\rangle = 0$ yields the output projected wavefunction as $1/2\,(|\uparrow;z\rangle + |\downarrow;z\rangle)$. After being measured in the final analyzer (C), we construct the projected wavefunction $1/2|\uparrow;z\rangle\,\langle z;\uparrow|(|\uparrow;z\rangle + |\downarrow;z\rangle) = 1/2|\uparrow;z\rangle$ for the state exiting the $+$ exit.

So the probability to emerge from the + exit of the analyzer (C) is the norm squared of the projected wavefunction or $1/4\,\langle z;\uparrow \mid \uparrow;z\rangle = 1/4$. The same probability occurs for exiting the −exit of the analyzer (C). One way of summarizing this behavior is to say that *the atom is stupid*—implying it only remembers the last axis it was projected onto. Hence, an atom entering with a positive vertical projection will then assume a horizontal projection after being measured on the x-axis and thereafter can be found to have a negative projection on the vertical axis if measured on the z-axis. This is because the atom cannot have a definite projection on the x- and z-axes at the same time. What about the total probability? If only 25% exit the + exit and 25% exit the −exit, we have lost 50% of the atoms. Indeed, we have, as those atoms emerged from the −exit of the x-oriented analyzer (B) and were ignored in the experiment.

Next, we describe the Stern-Gerlach analyzer loop. This device nominally splits the atomic beam according to its projection along the orientation of the analyzer loop and then rejoins it again. But there is no way for us to verify this behavior unless a measurement is performed (examples of possible measurements are given below), and so we prefer to say that the analyzer loop allows us to measure the projection of an atom in the analyzer loop orientation if we choose to or to leave the atom in its original state if we choose not to perform a measurement. (This issue is similar to the situation in a two-slit experiment where we do not know which slit the photon goes through or how it "interferes with itself" if we do not watch at the slits.) Because we created the Stern-Gerlach analyzer to pipe the atoms into horizontal beams at the exit, we can make the analyzer loop by attaching (magnetic) "plumbing" after the analyzer to recombine the two paths (see Fig. 4). As we will see below, we also could call this a "measurable basis-changer," but we stick with the original name from Styer.[4]

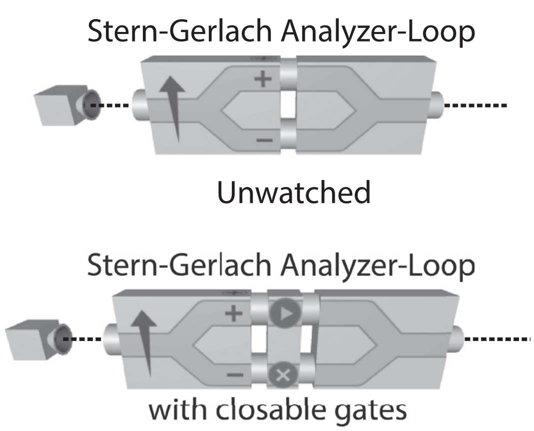

Fig. 4. Top: schematic of an analyzer loop, which can be thought of as a Stern-Gerlach analyzer attached to "plumbing" to recombine the two paths. If no measurement is made, then the analyzer loop does not alter the quantum state of the atom and it emerges with the same state it entered. If one of the paths is blocked, the atom emerges with the state given by the path that is not blocked (and the probability to emerge is determined by the state the atom had when it entered). Bottom: a Stern-Gerlach analyzer loop with a flow-through gate. The gate can be independently controlled to block zero, one, or two branches of the analyzer loop. The pictured flow-through gate is configured to block the lower branch of the analyzer loop (as indicated by ×). Blocking one path is a measurement. For example, if the atom entered in a state with a positive projection along the x-axis, half of the atoms would be blocked, and half would exit in the $\mid\uparrow;z\rangle$ state.

Instead of thinking of the analyzer loop as separating and rejoining the atomic beams, since this *is not a measurement*, the correct way to view the unwatched analyzer loop is that it places the atoms into a superposition of states according to the orientation of the analyzer loop. If no measurement is made, the original state that the atom entered with is unchanged. We describe this as the situation where *the analyzer loop does nothing*. If, on the other hand, we block one of the analyzer loop paths, then the atom is projected onto the state that *was not* blocked because a measurement was made that gave us which-way information. We describe the unwatched situation by saying that the action of the analyzer loop is to *change the basis* for the quantum state from whatever initial basis state the atom enters the analyzer loop into the basis corresponding to the axis oriented in the direction of the analyzer loop and then back to the original basis if no measurement is made.

For example, if the atom starts in an up-state along the z-axis and enters an unwatched analyzer loop oriented along the x-axis, then the atomic state can be thought of as initially being in the state $\mid\uparrow;z\rangle$, then being expressed in the x-basis as $1/\sqrt{2}\,(\mid\uparrow;x\rangle + \mid\downarrow;x\rangle)$ when the atomic beam "splits into two branches," and finally emerging as $\mid\uparrow;z\rangle$ after the "beams rejoined." Of course, this means *nothing happened to the atom* because the quantum state remained the same regardless of what basis it was expressed in. It is important to realize that the state does not collapse unless a measurement is made *inside* the analyzer loop (by blocking one path, for example). We feel this point is an important one to make with students because the notion of a state and the notion of the basis chosen to represent the state are often confused by students. The analyzer loop provides a unique opportunity to clearly describe this subtle distinction.

If, however, we watch at the branches with a device called a pass-through detector, shown in Fig. 5, then we are performing a measurement, and the results of the experiment will change. For example, consider the arrangement given in Fig. 5. The analyzer loop has a $\mid\uparrow;z\rangle$ state input. When an atom passes through one of the arms of the horizontal analyzer loop, it is measured by the pass-through detector. This corresponds to a projection onto the x-axis via $\mid\uparrow;x\rangle\,\langle x;\uparrow\mid$ when detected on the + branch or via $\mid\downarrow;x\rangle\,\langle x;\downarrow\mid$ when detected on the −branch. If we see an atom on the + branch, then we find the measurement due to the pass-through detector implies we have the projected wavefunction $\mid\uparrow;x\rangle\,\langle x;\uparrow \mid \uparrow;z\rangle = 1/\sqrt{2}\mid\uparrow;x\rangle$ emerge from the exit of the analyzer loop. Similarly, if the atom passes through the − branch, we have the down projected wavefunction along the x-axis.

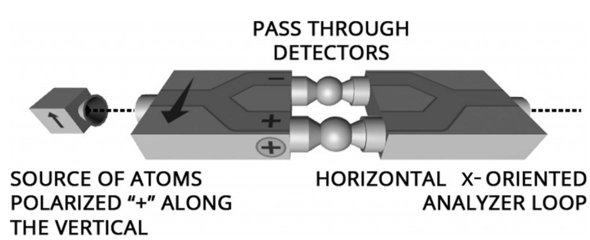

Fig. 5. Analyzer loop oriented along the x direction with pass-through detectors, which allow the path to be watched as the atoms move on the + or − branches. In this experiment, we have a polarized source of atoms producing the $\mid\uparrow;z\rangle$ state. We always see a full atom on one branch or the other. Watching the atoms changes their output state because it acts just like a measurement.

A subsequent measurement on a vertical Stern-Gerlach analyzer will produce an up spin half of the time and a down spin half of the time. This is completely analogous to the results from experiment 3 of Fig. 3 (bottom). So watching at the two branches is the same as measuring along them because it provides us with which-way information. Note that at no time do we see half of an atom going on two different paths. We always see a full atom on one path or on another path.

IV. DELAYED-CHOICE QUANTUM-ERASER STERN-GERLACH EXPERIMENT

We begin by re-iterating the quantum superposition effect of the analyzer loop. We start with an input atom in a definite state. The analyzer loop re-expresses the atom in a superposition of states according to the basis directed along the orientation of the analyzer loop, with no measurement. It then re-expresses the atomic state in the original basis as it emerges from the analyzer loop. This analog of quantum interference effects corresponds to the fact that the atoms all emerge in the same state they entered even though they were expressed as a superposition along a different axis when they were inside the apparatus. Since a basis change does not change the underlying quantum state, the unmeasured analyzer loop effectively does nothing to the atom.

We are now ready to start discussing the quantum eraser. The eraser works by first tagging the atoms via their internal quantum numbers, which may seem like it is a measurement when the atoms are on one of the two analyzer loop branches. But the tagging procedure still leaves the atoms in a pure superposition of quantum states, and so a measurement via a projection has not yet been made. For example, we assume there are two internal states, *unrelated to the spin of the atom*, which can be excited or de-excited. We attach an exciter to the + branch of an $\bar{x}$-oriented analyzer loop, as depicted in Fig. 6, right and denoted with the lightning bolt symbol. This device excites the internal structure of the atom from the ground state to the excited state without affecting the spin structure. This then can be employed to determine which path of the analyzer loop the atom takes simply by measuring the internal state of the atom.

Hence, tagging the atoms on the + branch by exciting them allows us to determine which-way information. We have correlated the internal state of the atom with the spin projection along the $\bar{x}$-axis by creating what one might want to call an "internally" entangled quantum state; we will refer to it as a tagged state so as to not confuse it with more conventional uses of entanglement. After tagging, we have a number of options available to us. If we measure the internal state, then we know which path the atom took through the analyzer loop, in direct analogy to what happened when we watched at the arms of the analyzer loop with pass-through detectors. But it is not exactly the same, because *we have not yet actually performed the measurement of the internal degree of freedom*. Our system has only been transformed to a tagged superposition of states at this stage. We must use a tensor product notation to describe this. We let $|ES\rangle$ denote the excited internal state and $|GS\rangle$ denote the internal ground state. Then, the exciter will take an input state of $|\uparrow;z\rangle \otimes|GS\rangle = 1/\sqrt{2}\,(|\uparrow;\bar{x}\rangle + |\downarrow;\bar{x}\rangle)\otimes|GS\rangle$ and transform it to $1/\sqrt{2}\,(|\uparrow;\bar{x}\rangle\otimes|ES\rangle + |\downarrow;\bar{x}\rangle\otimes|GS\rangle)$, which is a superposition corresponding to a pure (but tagged) quantum state. If we measure the internal state of the atom when it is in this superposition, we collapse the wavefunction and determine which branch the atom took through the analyzer loop—hence, we know the projection of its spin along the $\bar{x}$-axis, even though we did not directly measure the projection of the spin. Alternatively, if we measured the spin along the z-axis by passing through a vertically oriented analyzer (see Fig. 7), we would find half the time the atoms emerge as spin up and half the time as spin down, indicating we have which-way information about the paths in the $\bar{x}$-oriented analyzer loop. Furthermore, if we subsequently measure the internal state of the atom, we actually can determine which path the atom took through the $\bar{x}$-oriented analyzer loop, even after the spin state has been measured in the vertical analyzer!

The tagging phenomenon is a subtle one. While it is not a measurement, because it can still be erased, it is also not the same as if we did nothing. For example, we know that if the exciter was replaced by a pass through tube, then we would measure all atoms as being up when they exit the final analyzer. But after tagging, we recover the same results in the final detector as we would have if we did measure when in the horizontal analyzer (half up and half down). This occurs because the tagging has made the two paths distinguishable. Analyzing this situation requires an additional measurement postulate. If we view this from a quantum information perspective, we would say we must form what is called a partial

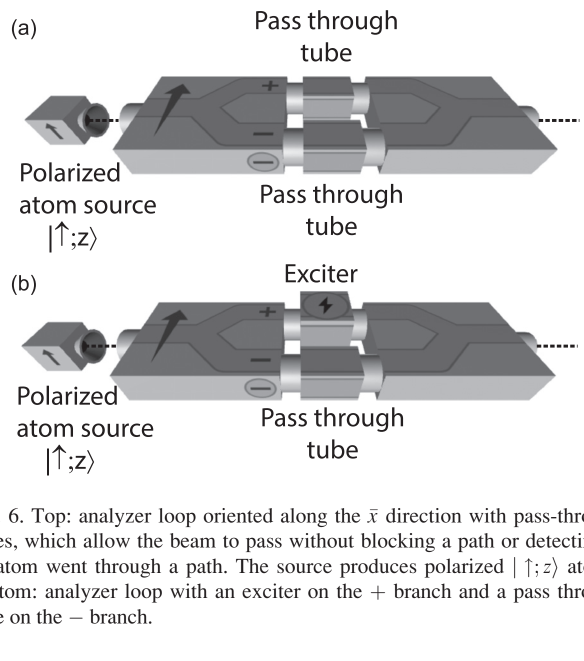

Fig. 6. Top: analyzer loop oriented along the $\bar{x}$ direction with pass-through tubes, which allow the beam to pass without blocking a path or detecting if an atom went through a path. The source produces polarized $|\uparrow;z\rangle$ atoms. Bottom: analyzer loop with an exciter on the + branch and a pass through tube on the − branch.

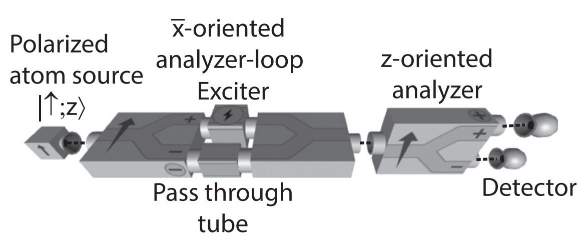

Fig. 7. "Unerased" version of a quantum eraser experiment with an analyzer loop-exciter (exciting on the + path of the $\bar{x}$-oriented analyzer loop) and a z-oriented analyzer to detect the spin projection of the atoms at the end of the experiment.

trace over the internal states of the atom because we are not measuring them—this produces a mixed state for the spin degree of freedom, producing half up and half down in the output of the final analyzer. If we instead view it more traditionally, this case corresponds to what is called a positive operator valued measurement (POVM)—here, the rule is that we add the probabilities for each distinguishable state, again producing half up and half down at the exit. (The terms partial trace, mixed state, and positive operator valued measurement are all common terms from quantum information, which are used to more precisely describe some of the subtle aspects of quantum measurements.) How much of this detail the instructor wants to relate to the students depends on how much they are likely to grasp. It is probably better to revisit this scenario later in a course, when concepts like density matrix, partial trace, and mixed states are introduced.

We successfully "untag" the atom if we can completely restore the atom to its initial state of $|\uparrow;z\rangle \otimes |GS\rangle$. We demonstrate shortly that this can be done only for half of the incident atoms in the system due to the complex nature of the tagging, which has correlated the internal degrees of freedom of the atom with the different spin states into a quantum superposition. Hence tagging, unlike watching, allows us the possibility to untag the atoms and restore the original state because tagging does not constitute a measurement. The untagging procedure is more commonly called a quantum eraser.

We pause for a moment to discuss the nomenclature we use of "tagging" versus "internal entanglement." While it is true that any set of quantum degrees of freedom that can be represented as tensor products can be employed to create entangled quantum states via superpositions, we prefer to use the word tagging to describe this procedure when we are forming superpositions between internal degrees of freedom of the same particle (similar to polarization of photons and their position or momentum) and reserve entanglement for the many-body entanglement of multiple particles in an entangled state, as in an EPR pair or a spin-singlet state formed from two spin-one-half fermions. We do not use any many-particle entangled states in this work.

We next extend the $\bar{x}$-oriented analyzer-loop with exciter experiment by having the analyzer-loop output go through a z-oriented analyzer loop, as shown in Fig. 7. We find that the exiting atoms emerge half of the time from the $+$ branch and half of the time from the $-$ branch. In addition, the atom will be in the ground state half of the time and in the excited state half of the time, *with no correlation between the spin state and the internal state after emerging from the vertical Stern-Gerlach analyzer.* Nevertheless, by measuring the internal state of the atom, we can immediately know whether it went through the $+$ or $-$ branch of the $\bar{x}$-oriented analyzer loop, even though we have "scrambled" the spin projection by measuring it on the z-axis.

Let's be sure we understand this by carefully going through the quantum analysis. We measure the probability to exit the $+$ exit of the z-oriented analyzer by projecting the output state of the analyzer loop onto $|\uparrow;z\rangle\langle z;\uparrow|$ and then finding the norm of the final projected wavefunction. Hence, this measurement produces

$$\frac{1}{\sqrt{2}}(|\uparrow;z\rangle\langle z;\uparrow|\uparrow;\bar{x}\rangle \otimes |ES\rangle + |\uparrow;z\rangle \times \langle z;\uparrow|\downarrow;\bar{x}\rangle \otimes |GS\rangle) = \frac{1}{2}|\uparrow;z\rangle \otimes (|ES\rangle + |GS\rangle). \quad (1)$$

The norm then becomes $1/4\langle z;\uparrow|\uparrow;z\rangle(\langle ES|ES\rangle + \langle ES|GS\rangle + \langle GS|ES\rangle + \langle GS|GS\rangle) = 1/2$ because the excited and ground states are orthogonal ($\langle ES|GS\rangle = \langle GS|ES\rangle = 0$). In addition, half of the time, the atom exiting the $+$ exit of the z-oriented analyzer will be in the ground state and half of the time in the excited state. The analysis for the $-$ exit yields identical final probabilities and final internal atomic states.

Next, we would like to erase the which-way information and restore the initial spin state the atom had before it entered the analyzer loop-exciter. In other words, we want to untag the tagged atoms. This requires two stages to work. First, we must have all atoms that emerge from the analyzer loop-exciter go through a superpositioner (graphically denoted with an S label). The superpositioner corresponds to what is called a Hadamard gate in quantum information and what is called a $\pi/2$ pulse in nuclear magnetic resonance; like the exciter, it does not perform a measurement. We call it a superpositioner because it corresponds to half of the exciter operation—it creates a superposition of ground and excited states. In other words, it transforms the ground state to the superposition $|GS\rangle \to 1/\sqrt{2}\,(|GS\rangle + |ES\rangle)$ and it transforms the excited state to the superposition $|ES\rangle \to 1/\sqrt{2}\,(|GS\rangle - |ES\rangle)$. Because these two states remain orthogonal to each other (and hence completely distinguishable), we can still tell them apart, and so the superpositioner does not erase the which-way information. We simply need to measure the atomic states in the appropriate basis, since measuring just $|GS\rangle$ or $|ES\rangle$ will not be able to provide the which-way information. Note that the superpositioner does change the quantum state. This is the difference from a simple change of basis, which preserves the quantum state.

The which-way information is finally erased by measuring only the atoms in the ground state. This is accomplished by employing a de-exciter (denoted by the electrical "ground" symbol). The de-exciter will force the excited state to transition to the ground state and emit a photon, but it does nothing if the atom enters it in the ground state. If a photon is detected, the de-exciter then blocks the atom and does not allow it to exit. Hence, the de-exciter acts like a pass-through filter, which only allows atoms that entered it in their ground-state to pass through; hence, an equivalent name would be "ground-state filter." We can perform this measurement any time before the atom enters the final z-oriented analyzer or any time after the atom has emerged from an exit of the z-oriented analyzer (see Fig. 8). This allows us to make a delayed choice for whether we erase the (tagged) quantum information or not. The choice can be made *after all other measurements have been completed!* To be clear, the de-exciter does perform a projective measurement, but only on the internal state of the atom, not on the spin. Nevertheless, it does collapse the wavefunction.

The quantum analysis including the de-exciter is completed as follows: begin with the state emerging from the analyzer loop-exciter, given by $1/\sqrt{2}\,(|\uparrow;\bar{x}\rangle \otimes |ES\rangle + |\downarrow;\bar{x}\rangle \otimes |GS\rangle)$. After passing through the superpositioner, this state becomes

$$\frac{1}{2}[(|\uparrow;\bar{x}\rangle + |\downarrow;\bar{x}\rangle) \otimes |GS\rangle + (-|\uparrow;\bar{x}\rangle + |\downarrow;\bar{x}\rangle) \otimes |ES\rangle]. \quad (2)$$

Next, we re-express this quantum state in the z-basis for the spin, instead of the $\bar{x}$-basis. This yields $1/\sqrt{2}\,(|\uparrow;z\rangle \otimes |GS\rangle + |\downarrow;z\rangle \otimes |ES\rangle)$. Hence, we have shifted the entanglement

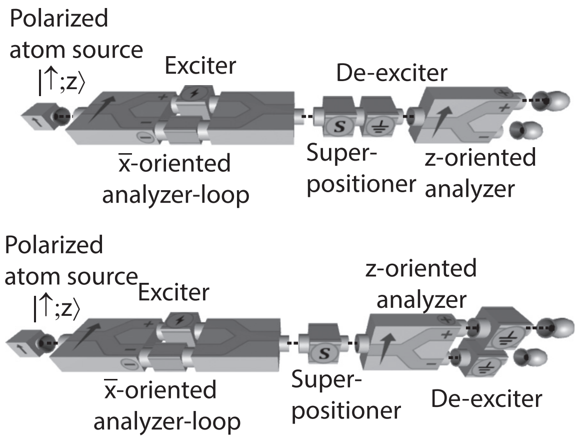

Fig. 8. Fully erased quantum eraser experiment with the eraser elements (superpositioner and de-exciter) either both positioned before the final analyzer (top) or one before and the other after (bottom). In the second case, the de-exciter can be placed as far from the analyzer as desired. Note how in the latter case, atoms emerge from both exits of the z-oriented analyzer, but only those that are in the $|\uparrow\rangle \otimes |GS\rangle$ state can pass through the de-exciter and be detected. The delayed choice corresponds to whether the de-exciter is inserted or not; we have pictured the case where it is inserted.

to now be the superposition of an up spin along the z-axis correlated with the ground state and the down spin along the z-axis correlated with the excited state. We continue the analysis for the scenario depicted in the bottom of Fig. 8. Measuring in the z-oriented analyzer requires the projections onto the $|\uparrow; z\rangle$ or $|\downarrow; z\rangle$ states, respectively. We find half the time, the atom emerges in the $|\uparrow; z\rangle \otimes |GS\rangle$ state and half the time in the $|\downarrow; z\rangle \otimes |ES\rangle$ state. Now, if we decide to record the measurements only for atoms that emerge from the de-exciter (that is, entered the de-exciter in the ground state), we remove all $|\downarrow; z\rangle \otimes |ES\rangle$ atoms. This has then erased the which-way information, and we find the atom emerges from the quantum eraser with the same state it first entered the analyzer loop, namely, the positive projection of spin along the z-axis in the ground state!

Note that we lose half of the atoms when we do this. This behavior is typical of quantum eraser measurements. We must remove the atoms that have the wrong quantum behavior, and hence, we lose signals when we restore the original quantum coherence that we lost by tagging the system to allow us to determine the which-way information. While, in principle, one might be able to devise a clever way to overcome this issue by using interaction-free measurements, it appears to be an issue with all quantum eraser measurements. The full quantum state is not restored by the eraser because we must remove the "bad" measurements from the experiment. Note, on the other hand, if we do not measure the internal state of the final atom, then we find half of the atoms emerge from the + exit and half from the − exit of the z-oriented analyzer. This is exactly what happens when the atoms are watched or whenever we have which-way information.

Wheeler originally suggested[11,12] that perhaps the delayed choice measurement implies that the quantum particles infer their behavior by moving backwards in time. But we see this is not necessary at all when one performs a careful quantum analysis. Indeed, the eraser works by carefully manipulating the correlations and entanglement between the different quantum states of the quantum particle (ground or excited state and spin). Similarly, in a two-slit experiment it arises from which slit the photon went through and its polarization. Hence, all of the information is in the linear combinations of tensor products of the wavefunction, and that is all one needs to understand and analyze these experiments.

There are a number of variants one can include for further discussion or as problems for the students. These include the following possibilities: (1) change the orientation of the analyzer loop from a horizontal direction to a different angle with respect to the vertical such as 45°; (2) place the de-exciter in front of the final z-oriented analyzer, so that all of the atoms that emerge from the final analyzer are ground-state atoms in the + state along the z-axis; (3) allow the students to complete the delayed choice analysis on their own instead of doing it for them, and (4) have the students discuss whether the superpositioner could be placed after the z-oriented analyzer but before the de-exciter.

In addition to providing a neat exercise in working with tensor-product states, the analysis of the delayed choice Stern-Gerlach quantum eraser allows the students to fully understand a complex experiment with a rather elementary analysis, which requires applying just a few quantum rules. When coupled with videos of the quantum eraser for the two-slit experiment, this can be a powerful way to help students understand quantum phenomena early in the curriculum and to build confidence that this material can be understood easily if one simply analyzes the behavior according to the quantum rules.

V. POSSIBLE IMPLEMENTATION IN A REAL ATOMIC SYSTEM

We briefly describe how one might actually perform such an experiment in a lab because we believe it enriches the discussion if the experiment has a possibility to actually be realized. The main challenge with implementing the delayed choice Stern-Gerlach quantum eraser in a real system is that the transition between the internal states of the atom must not change the total electronic angular momentum of the system, which determines the projection of the angular momentum onto the axis of the Stern-Gerlach device. Electronic transitions between different atomic energy levels are likely to affect such states as the total angular momentum usually changes for these transitions. Furthermore, such excited states are very short-lived (few nanoseconds to microseconds) and would not survive long enough for an experiment to be completed.

Instead, we propose to perform experiments with the ^{171}Yb atom, a species known to enable an ultra-accurate optical frequency atomic clock.[37] The Ytterbium atom has two J = 0 atomic clock states, the $^1S_0$ and the $^3P_0$ states, each of which has angular momentum zero. The ^{171}Yb isotope also has a nuclear spin one-half and can be prepared and detected in either its positive or negative projection states. Although the $^1S_0 \rightarrow {}^3P_0$ clock transition near a laser wavelength of 578 nm is strictly forbidden, the presence of the nuclear spin breaks the symmetry and permits laser excitation to the excited state, so that any superposition of ground and excited states could be prepared in the atomic clock experiment. Since the coupling of these $J = 0$ electronic states to the nuclear spin is extremely weak, the excited state lifetime is quite long, and the nuclear spin constants are nearly the same in the ground and excited states. Thus, the electronic and nuclear spin degrees of freedom can be taken as essentially independent.

While one might think that the ^{171}Yb atom provides a nearly ideal system to realize our various Stern-Gerlach schemes, there is one problem. The nuclear magnetic moment,[38] 0.49367 μ_N for ^{171}Yb, is much smaller than the electron magnetic moment used for a typical Stern-Gerlach separation of spin states. Electronic magnetic moments are on the order of one Bohr magneton ($\mu_B/\hbar = 14.0$ GHz/T), whereas the nuclear magneton ($\mu_N/\hbar = 7.62$ MHz/T) is nearly 2000 times smaller. The original experiment of Stern and Gerlach used a beam of silver atoms, which have a single unpaired electron. They were able to separate the two electronic spin projections by several tenths of a millimeter using a quite strong field gradient of a few T/cm. Thus, achieving practical separations with a small nuclear magnetic moment requires impractically large magnetic field gradients. This certainly creates a challenge with implementing such an experimental system in practice, but it does show that in principle, such a system can be used in these experiments.

It may be possible to use the optical Stern-Gerlach (OSG) effect to achieve large enough separations in order to implement our scheme. The separation of nuclear spin components using the OSG method has already been demonstrated[39] with ^{171}Yb and ^{173}Yb and the similar atomic clock species (Ref. 40) ^{87}Sr. The latter species has a nuclear spin of 9/2, which could be separated into 10 separate spin projection states using the OSG effect with ultracold atoms. The optical separation is based on using the strong light intensity gradient in a focused laser beam to separate the different spin components, which couple differently to the laser field and experience differential optical forces. Whether a practical OSG experiment could be designed for our scheme would need to be carefully considered, since the ground and excited electronic states do not in general experience the same optical forces, although it is often possible to find "magic wavelengths" where they are the same.

One should also note that it can be quite challenging to create the analyzer loop, as discussed in the so-called "Humpty-Dumpty" series of papers,[41–43] since one needs to maintain the magnetic fields to a high level of tolerance and some decoherence is almost certainly going to occur. It is not clear, however, whether this also holds in the situation where one creates the Stern-Gerlach experiment optically, as we proposed here.

VI. APPLICATION TO OTHER EXPERIMENTS

One of the most common examples of a delayed choice quantum eraser is to perform the two-slit experiment with crossed polarizers over the slits and a polarizer that is employed at the screen before measuring the pattern of light.[27] If the polarizers at the slits are horizontal and vertical, respectively, then a horizontal polarizer at the screen will see a single slit pattern, as will a vertical polarizer. But if the polarizer at the screen is rotated to 45°, then the interference pattern emerges. Numerous YouTube videos of this experiment exist, and it can be implemented rather easily at home using just a laser pointer and polarizers from 3D movie glasses.[14]

Because this paper is focused on the Stern-Gerlach experiment, we do not go through the full analysis of the conventional two-slit experiment here, but it should be clear that a quite similar analysis can be done of this experiment, and it reinforces the concepts covered for the Stern-Gerlach experiment. Depending on when one wants to discuss polarization in the quantum mechanics class, this might come later in the curriculum than the Stern-Gerlach experiment.

In addition, the same techniques employed here for the delayed choice Stern-Gerlach experiment can also be employed to examine other interesting experiments, as Styer does in his text.[4] These include a modified version of the Einstein-Podolsky-Rosen experiment and of the Bell experiments. We feel including all of these additional topics greatly enhances the undergraduate quantum curriculum and would not take too much time away from more standard topics. We feel the benefits that the student gains from having contact with modern quantum experiments and from understanding concepts such as the superposition and measurement in a more concrete fashion far outweighing the cost in time to other subjects that might need to be dropped from the course.

VII. CONCLUSIONS

As more and more quantum classes embrace the Stern-Gerlach-first curriculum, it becomes possible to employ this experiment to cover a range of interesting modern quantum experiments that showcase the fascinating nature of quantum mechanics while strengthening the students' abilities in understanding concepts such as the superposition, tensor products, and measurement. Tackling these concepts early on will help ground the students in the fundamentals of quantum mechanics and better prepare them for the rest of the quantum curriculum they will cover in their course. Given the fact that they already have all of the prerequisite knowledge needed from current textbook coverage of the Stern-Gerlach experiment, the extension we have described provides students with an easy entry into more sophisticated material. We hope other quantum mechanics instructors will agree.

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