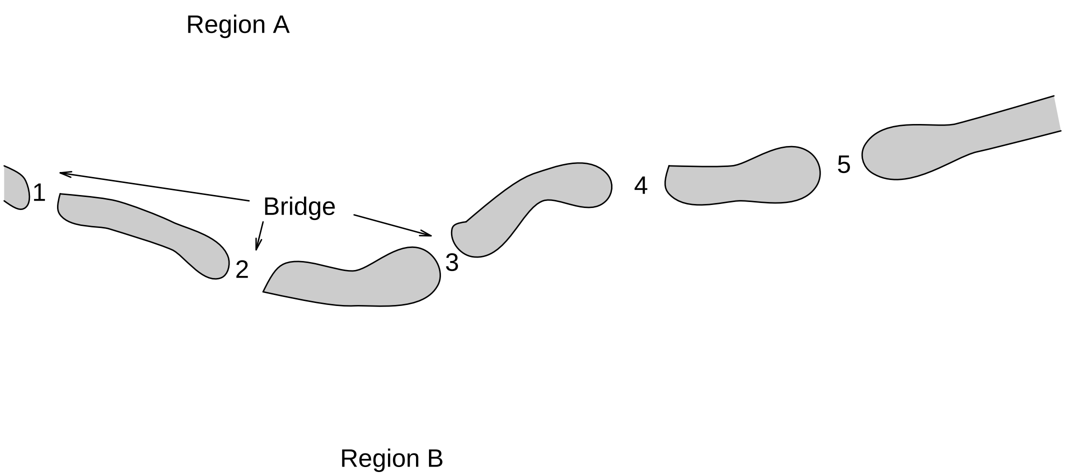
Physics 008—Tutorial 5: The Königsberg Bridge Problem

1. Odd Number of Bridge Crossings



Suppose we simplify the Königsberg bridge problem so that there are only two regions A and B that are connected by bridges. If we only have **one** bridge connecting the two regions, then the only possible paths that cross the bridge once are AB and BA (depending on which region we start our walk from). In either case, A appears once in the path, and there are always a total of two letters in the path.

What happens if there are **three** bridges? What is the total number of letters in a path that crosses each bridge once? How many times will A occur in the path? (Drawing a figure and some paths should help you answer this.) Does it matter whether the path starts in region A or in region B?

Now suppose there are **five** bridges. What is the total number of letters in a path that crosses each bridge once? How many times will A occur in the path?

What happens if there are **seven** bridges?

Generalize your results above to formulate a rule for the total number of letters in a path that crosses each bridge once, if there are n bridges leading into region A, where n is an odd number.

Generalize your results above to formulate a rule for the number of times A appears in that path.

2. Even Number of Bridge Crossings

The case with an even number of bridges is little more complicated. Consider first the case of **two** bridges. How many total letters appear in the path? How many times does A appear in the path if we start in region A? (Once again drawing a figure should help you.)

What about if we start in region B?

Now consider what happens with **four** bridges. How many total letters are there in a path? How many times does A appear in the path if we start in region A? How many times if we start in region B?

Finally consider **six** bridges. How many total letters in the path? How many times does A appear in the path if we start in region A? How many times if we start in region B?

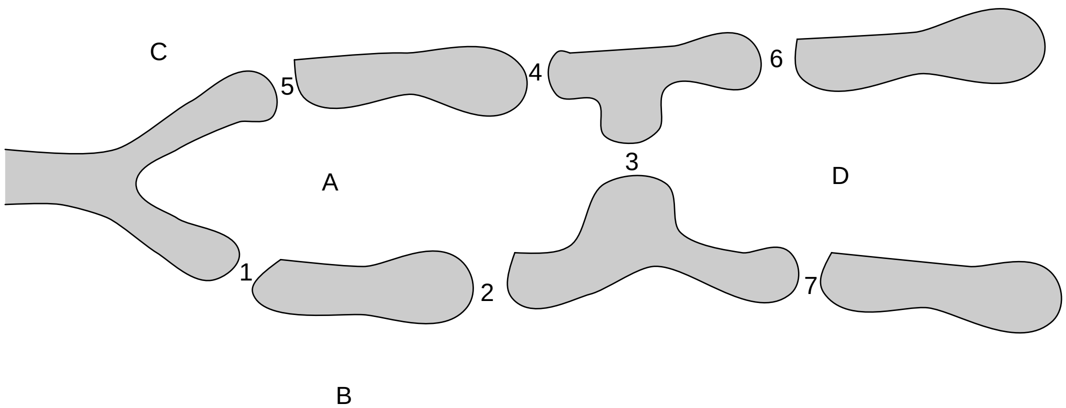
Formulate a general rule for the total number of letters in a path when there are n bridges leading into region A (n is an even number). (Does it matter whether you start in region A or in region B?)

Formulate a general rule for the number of times A must appear in the path when there are n bridges leading into region A (n is an even number) and the path starts in A.

Formulate a similar rule that holds when the path starts in B. Make sure your results from above agree with your general rules.

Check your results with an instructor before proceeding.

3. The Königsberg Bridge Problem



The river Pregel is in gray. Seven bridges cross into the four regions.

We turn now to the Königsberg Bridge Problem, illustrated above. We will use Euler's general procedure outlined below. Complete the table on page 4 as we proceed through each step.

1. First designate the regions separated by the water by letters such as A, B, C, etc. This has been done for you in the figure above.

2. What is the total number of letters that can appear in a path that crosses each bridge only once (hint: consider your answers on page 1 and 2)? Write that number above the table .

3. Write the letters A, B, C, etc. in a column, and note next to the letter how many bridges enter the region.

4. Place an asterisk * next to each region that has an even number of bridges entering it.

5. In the third column write the number times each region appears in a path that crosses each bridge exactly once. For a region with an even number of bridges, assume that the path **does not** start in that region.

6. Add up the numbers in the last column. If the sum is greater than the number written at the top, then the journey cannot be made.

Use these rules to complete the following table for the Königsberg bridge problem.

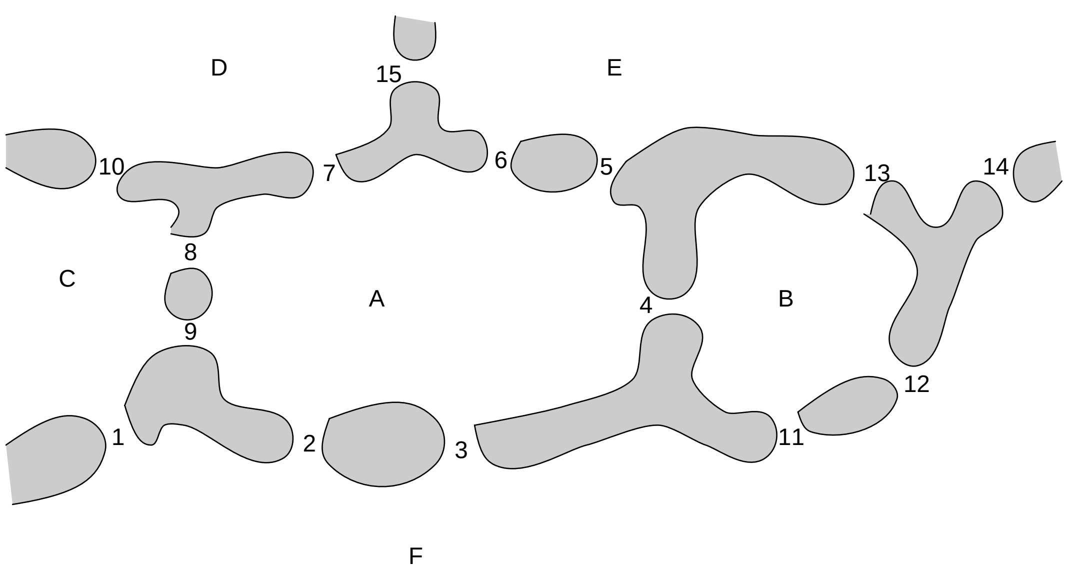
$$\text{Number of bridges} + 1 = 7 + 1 = 8$$
$$= \text{total number of letters in path}$$

Region	Number of Bridges	Number of Times Region Must Appear in Path
A	5	3 $[= \frac{1}{2}(5+1)]$
B		
C		
D		
	total	

Is it possible to construct a continuous path that crosses each bridge exactly once? Explain why or why not. If it is possible, sketch such a path on your figure.

4. Another Bridge Problem

Construct a table for the following bridge problem:



Number of bridges + 1 =

Region	Number of Bridges	Number of Times Region Must Appear in Path
A		
B		
C		
D		
E		
F		
	total	

Don't forget to include the asterisks on the relevant regions. Can these bridges be traversed crossing each one only once? If so, from which region(s) can such a path begin? Sketch such a path on your figure if it is possible.

5. Final Rules

If the sum of the number of times a region must appear in a path is equal to or is one less than the number at the top of the table (one plus the total number of bridges), the journey is possible. Suppose the sum is equal to the number at the top of the table. Which type of region must the path start in? Explain.

If the sum is one less than the number at the top of the table, which type of region must the path start from?

Euler simplified these rules to the following (see the Scientific American article for details):

- If there are more than two regions approached by an odd number of bridges, no route satisfying the required conditions can be found.
- If, however, only two regions have an odd number of approach bridges, the journey can be completed if it starts in one of those regions.
- If, finally, there are no regions with an odd number of bridges, the required journey can be made, no matter where it begins.

Explain why you cannot have only one region with an odd number of bridges.

Are these simplified rules consistent with your results for the last two bridge problems?