Refractive index measurement using total internal reflection

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An undergraduate level experiment is presented for demonstrating total internal reflection. The experiment yields a relatively precise value of the refractive index of a prism to be found from a measurement of the incident angle at which total internal reflection occurs. Analytic expressions are obtained for the refractive index as a function of the incident angle for both equilateral and right angle prisms. Students can observe not only the critical angle at which total internal reflection occurs, but measure and analyze the significant rise in reflected beam power. The addition of a small fluid reservoir attached to the side of the prism enables the determination of a liquid refractive index. © 2005 American Association of Physics Teachers.

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I. INTRODUCTION

Laboratory-based courses are critical for undergraduate students to develop skills needed in both industry and academia. The experiments in these courses must both engage students and enable their development of skills such as error analysis. This paper presents an experiment that can be implemented in an upper level undergraduate optics course for the measurement of refractive indices using total internal reflection.

The experiment demonstrates several aspects of total internal reflection and the refraction of light. Light incident on an equilateral or right angle prism is refracted through the prism, strikes the second surface, and is further refracted and reflected. As the prism is rotated with respect to the incident beam, the ray incident on the second surface reaches the critical angle for total internal reflection. At this angle, the transmitted ray at this interface is parallel to the prism surface and, at larger incident angles, the beam is completely reflected.

In the second part of the experiment, a liquid reservoir in contact with the surface at which total internal reflection occurs is filled with water, and the known prism refractive index is used to find the refractive index of the liquid. A simple expression for the refractive index in terms of the incident angle for total internal reflection also is found.

In addition to the determination of the refractive indices, the experiment introduces the phenomenon of total internal reflection, which has several current applications, including microscopy of cell membranes attached to surfaces, fiber optic sensors, and fingerprint identification.

II. EXPERIMENTAL PROCEDURE

The incident beam can be provided by a small continuous laser. A stabilized 10 mW HeNe laser (Spectra Physics) was used. Either equilateral or right angle prisms may be used. The initial measurement is the determination of the prism material refractive index from the incident angle at which total internal reflection occurs. In our setup both the laser and the prism were positioned on an optical breadboard (Edmund Industrial Optics) using standard optical mounts (posts and post holders). The breadboard and mounting hardware were used for ease of setup, but are not necessary, because the experiment is not significantly sensitive to vibrations. Excluding the breadboard, the cost of all the equipment, including the laser and rotary stage, is roughly $500. A small laser power meter for the measurement of reflected power versus angle costs $\approx$300.

The prism is best mounted on a rotary stage for ease of varying the angle of incidence. In our measurements, a rotary assembly with a prism holder from Edmund Industrial Optics was used. If a rotation stage with angle gradations is available, the beam that is reflected directly back from the prism surface to the laser can be used to find normal incidence, and the change in the angle as the prism is rotated is observed. The angle between the incident beam and the normal to the first (incident) surface of the prism is measured when total internal reflection occurs, that is, when the beam exiting the second surface of the prism is parallel to that surface. However, a simpler and generally more precise value can be found by measuring the position of the back-reflected spot and using its location with respect to the point of incidence on the prism to determine the angle (see Fig. 1).

By measuring the back-reflected spot at a large distance from the prism, a precision of tenths of degrees can be obtained. For example, in our setup the distance between the prism and ruler was 1.0 m, and the laser spot position was determined to within 1 mm, giving a precision of $<0.06^\circ$ in the change of the angle from the set point, which was $45^\circ$ in our case.

The limit on the accuracy of the determination of the absolute value of the angle is set by the accuracy of the set point determination. We used a T-square to set the zero point at an incident angle of $45^\circ$ (beam reflected at $90^\circ$) with an accuracy of $\sim0.1^\circ$. These estimates of the measurement accuracy were confirmed by the extracted values of the refractive index for prisms of known refractive index. The precision is particularly important in the measurement of the reflection coefficient $r$ near the critical angle, because $r$ changes by over 50% within a few degrees of the critical angle.

This angle is measured when total internal reflection occurs, that is, when the beam exiting the second surface of the prism is parallel to this surface. A major source of confusion is determining exactly this beam because several beams corresponding to multiple reflections inside the prism may be observed exiting the second surface of the prism. As a guide, the angle can be first calculated with an estimate of the refractive index. For a typical high refractive index prism ($n_g$...
1.7) total internal reflection occurs at an angle close to 45° between the incident beam and the normal to the first prism face.

The second part of the experiment is the determination of a liquid refractive index. A small reservoir is machined from plastic and affixed to one side of the prism, as shown in Fig. 2. When filled with a liquid, the reservoir presents a different interface for total internal reflection. Once the refractive index of the prism is known, the refractive index of the liquid can be determined.

A more accurate determination of the refractive index can be achieved by measuring the power of the beam reflected from the second (total internal reflection) surface of the prism as a function of the incident angle. This beam is reflected to the third prism surface, where it is partially transmitted. The ratio of the reflected power to the incident power can be used to find the reflection coefficient from the second surface of the prism, where total internal reflection occurs. We used a calibrated laser power meter (Metrologic model 45-545). However, because only the ratio of the power for the incident and reflected beams is needed and not the absolute value of the power, any small photodetector will work well. The detector must be moved relative to the prism, or a positive lens may be used to maintain the beam incident on the detector. The value of the reflection coefficient depends on the polarization of the incident beam with respect to the scattering plane. A linear polarizer placed at the laser output will allow the polarization to be fixed; however, many unpolarized HeNe lasers display slow fluctuations in the output polarization state, even when the total power is relatively constant, in particular if they operate in more than one longitudinal mode. This problem can be resolved by splitting off part of the beam and referencing the reflected beam power to that of the split-off part.

III. THEORY

The angle of reflection is quantitatively described by Snell’s law,

\[ n_1 \sin \theta_1 = n_2 \sin \theta_2, \quad (1) \]

where \( n_{1,2} \) and \( \theta_{1,2} \) are the refractive index and angle of incidence on either side of the interface, and the angles are measured with respect to the normal. At the critical angle, \( \theta_2 = 90^\circ \),

\[ \theta_{1,\text{crit}} = \sin^{-1} \left( \frac{n_2}{n_1} \right). \quad (2) \]

We use Eqs. (1) and (2) to determine the refraction angle at the incident interface (air to glass) and the second interface (glass to air or water). Simple geometry gives the relation between the two internal angles. For both the equilateral and right angle prisms, a simple expression can be derived for the refractive index as a function of the incident angle for light entering the prism at which total internal reflection occurs.

The application of Eq. (1) at the first (incident) interface gives

\[ \sin \theta_1 = n_2 \sin \theta_2, \quad (3) \]

where all angles are measured with respect to the normal as defined in Fig. 3, \( n_g \) is the refractive index of the prism glass, and \( n = 1 \) is assumed for air. For an equilateral prism, the angle at which the beam is incident on the second surface can be shown using simple geometry to be

\[ \theta_3 = \frac{\pi}{3} - \theta_2. \quad (4) \]

At the critical angle, refraction at this interface gives

\[ \sin \theta_{3,\text{crit}} = \frac{1}{n_g}. \quad (5) \]

If we combine Eqs. (4) and (5) and use several trigonometric identities, we obtain...
\[
\frac{\sqrt{3}}{2} \cos \theta_2^{rit} - \frac{1}{2} \sin \theta_2^{rit} = 1
\]

The result of combining Eqs. (6) and (1) is

\[
\frac{\sqrt{3}}{2} \sqrt{n_g^2 - \sin^2 \theta_1^{rit}} - \frac{1}{2} \sin \theta_1^{rit} = 1
\]

which yields a simple expression for \( n_g \) in terms of the critical angle of incidence on the prism \( \theta_1^{rit} \):

\[
n_g^2 = \frac{4}{3} \left( 1 + \sin \theta_1^{rit} + \sin^2 \theta_1^{rit} \right).
\]

The error in \( n_g \) due to the uncertainty in the value of the critical angle can be found from the error propagation equation for the standard deviation:

\[
\sigma_{n_g} = \left[ \frac{\partial n}{\partial \theta} \right]_{\theta = \theta_1^{rit}} \frac{1}{2} \left[ \cos \theta_1^{rit} + 2 \cos \theta_1^{rit} \sin \theta_1^{rit} \right] \sigma_{\theta_1}.
\]

In Sec. IV we estimate an uncertainty of \( \sim 0.20^\circ \) (3.5 \( \times 10^{-3} \) rad); for a typical critical angle of 45°, we find an uncertainty of 0.001 in the refractive index.

A similar derivation can be carried out for total internal reflection from a right angle prism. The corresponding expression for \( n_g \) as a function of the angle of rotation of the prism is

\[
n_g^2 = 2(1 + \sqrt{2} \sin \theta_1^{rit} + \sin^2 \theta_1^{rit}).
\]

where the beam is incident on one of the smaller prism faces, and total internal reflection occurs on the large prism face.

IV. REFRACTIVE INDEX OF SF18 GLASS

By using the displacement of the back-reflected beam from the incident face of the prism, the angle of the prism can be determined with a high precision. Both the distance from the prism to the ruler and the position of the back reflected beam can be measured with precision of 1–2 mm, yielding a precision of less than 0.1° in the angle. However, several other sources of error increase the uncertainty in the determination of the refractive index. The center of rotation of the prism should be at precisely the point at which the beam is incident on the first surface. An offset of a few millimeters in this position will lead to an uncertainty in the baselines for the angle calculation, and a similar error of 0.1°. The laser beam is not a perfect plane wave; its divergence will result in a range of incidence angles. Typical HeNe lasers have divergence angles on the order of mrad. However, the largest uncertainty arises from the difficulty in determining the precise onset of total internal reflection. This determination is a somewhat subjective on exactly where the reflected beam can be measured with precision of 1–2 mm, yielding a precision of less than 0.1° in the angle. However, several other sources of error increase the uncertainty in the determination of the refractive index. The center of rotation of the prism should be at precisely the point at which the beam is incident on the first surface. An offset of a few millimeters in this position will lead to an uncertainty in the baselines for the angle calculation, and a similar error of 0.1°. The laser beam is not a perfect plane wave; its divergence will result in a range of incidence angles. Typical HeNe lasers have divergence angles on the order of mrad. However, the largest uncertainty arises from the difficulty in determining the precise onset of total internal reflection. This determination is a somewhat subjective on exactly where the refracted beam at the second prism interface disappears, and is a systematic error that is difficult to quantify precisely. From repeated measurements with multiple observers according to the same procedure as we have described, we found a range of roughly ±0.2°, which yields a refractive index accurate to three decimal places.

A set of ten measurements on an equilateral prism made of SF18 glass yielded an average value of 1.715 for the refractive index at \( \lambda = 632.8 \) nm. The refractive index of the prism material can be calculated using the Sellmeier equation, which is an approximate relation for the wavelength dispersion of the refractive index of a dielectric medium. From Ref. 7 we calculated for SF18 at \( \lambda = 632.8 \) nm the value \( n_g = 1.717 \), a difference of 0.002 from the measured value. The experimental procedure can be repeated with other quasi-monochromatic light sources, for example, red and green laser diodes and the results compared to the Sellmeier calculations. Polychromatic light sources also may be used, but the refractive index will be an average of the values in the wavelength range of the light.

V. DETERMINATION OF THE LIQUID REFRACTIVE INDEX

Once the refractive index of the glass has been determined, it can be used to find the index of a liquid in contact with the back face of the prism. As shown in Fig. 2, the prism is modified by attaching a small open reservoir to one face. Equation (5) must be modified to include the refractive index of the liquid in contact with the second interface, \( n_l \),

\[
\sin \theta_1 = \frac{n_l}{n_g}.
\]

If we combine Eqs. (1), (4), and (11) as before and solve for the unknown \( n_l \), we find

\[
n_l = \frac{\sqrt{3}}{2} \sqrt{n_g^2 - \sin^2 \theta_1^{rit}} - \frac{1}{2} \sin \theta_1^{rit}.
\]

For water in the reservoir and a prism refractive index of 1.73, the angle at which total internal reflection is observed is \( \sim 12.9^\circ \). We found a value of \( n_l = 1.331 ± 0.001 \) for the index of distilled water, equal to the published value\(^6\) of 1.3317 within the error of the experiment.

VI. REFLECTED POWER NEAR THE CRITICAL ANGLE

An effect that is easily seen with the naked eye is the sharp increase in the reflected power near the critical angle. Below the critical angle for total internal reflection, the incident power is split into transmitted and reflected parts. As the beam nears the critical angle, the transmitted power rapidly drops to zero, with a corresponding increase in the reflected power. In our experimental setup, this reflection is observed as the beam exiting from the third surface of the prism, as shown in Fig. 3. The reflected power versus incident angle \( \theta_1 \) can be determined from the Fresnel coefficients at all three interfaces: the transmission at the incident and exit surfaces, and the reflection at the second surface. The reflection coefficients are polarization dependent; for incident polarization perpendicular to the plane of incidence, the transmission coefficient at the first interface is\(^9\)

\[
t_\perp = \left| \frac{E_r}{E_i} \right|_\perp = \frac{2 \cos \theta_1}{\cos \theta_1 + n_g \cos \theta_2}.
\]

where the definitions of the angles are shown in Fig. 2.

At the second interface, the reflection coefficient is

\[
r_\perp = \left| \frac{E_r}{E_i} \right|_\perp = \frac{n_g \cos \theta_3 - \cos \theta_4}{n_g \cos \theta_3 + \cos \theta_4}.
\]

The transmission through the exit interface is

\[
t_\perp = \left| \frac{E_r}{E_i} \right|_\perp = \frac{2n_g \cos \theta_5}{n_g \cos \theta_3 + \cos \theta_6}.
\]
Similar expressions can be found for polarization parallel to the plane of incidence. The relative reflected and transmitted power at each interface are given by

\[ R_{i\perp} = \left( \frac{P_i}{P_i'} \right)_{\perp} = r_{i\perp}^2, \tag{16a} \]

\[ T_{i\perp} = \left( \frac{P_i}{P_i'} \right)_{\perp} = t_{i\perp}^2 \left( \frac{n_i \cos \theta_i}{n_i \cos \theta_i} \right), \tag{16b} \]

where the subscripts \( i \) and \( t \) refer to the incident and transmitted media, respectively. The additional term for transmission in the parentheses of Eq. (16b) accounts for the change in the beam cross section in the transmission medium due to refraction.

The total reflected power exiting the prism from the third surface is the product of the reflection and transmission factors in Eq. (16) for the three interfaces. Figure 4 shows typical data for the reflected power versus the incident angle (the angle of the light striking the first prism surface with respect to the normal) for both parallel and perpendicular polarizations for the prism only (no liquid). The data points are averages of five measurements, and the error bars indicate the statistical error. Both the parallel and perpendicular polarization data are slowly varying functions of the angle of incidence on an interface, except near the critical angle. The solid lines in the plot are fits of Eqs. (13)–(16) to the data, found by using a nonlinear iterative fitting procedure (based on a generalized reduced gradient method) with the refractive index of the prism the only fit parameter. A simultaneous fit to both sets of data (parallel and perpendicular) gives a value for the refractive index of 1.717, a difference of 0.0005, or 0.03% from the actual value. Commercial instruments such as Abbe refractometers, which can be used on both glasses and liquids, typically achieve resolutions one to two orders of magnitude higher (four or five decimal places).

Although the fit to the parallel polarization data is reasonable, the fit to the perpendicular polarization is poor. The deviations of the data from the curve are likely due to the antireflection coatings on the prism (\( \lambda/4 \) coatings of MgF\(_2\)), which have no effect on the angle for total internal reflection, but modify the reflection and transmission coefficients, especially at the input interface. Because the transmission coefficient at this first interface is near unity for parallel polarization, the antireflection coating will have little effect. However, for perpendicular polarization the transmission coefficient in this range of angles is on the order of 0.85 without the antireflection coating; with the coating, the total reflected power will be greater.

A similar set of measurements was done with the cell filled with water. A fit to the reflected power versus incident angle, assuming the previous value of the refractive index for the prism, gave a value of 1.3308 for the refractive index of water, a value less than 0.1% different than the published value of 1.3317 (at room temperature and wavelength 632.8 nm).\(^8\)

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6Joseph H. Simmons and Kelly S. Porter, Optical Materials (Academic, San Diego, 2000), pp. 103-104. The Sellmeier coefficients indicate the location and strength of electronic transitions that most strongly affect the refractive index.

7The Sellmeier coefficients for SF18 are \( A_1 = 1.564414 \, 36, \quad A_2 = 0.291413 \, 58, \quad A_3 = 0.960307 \, 88, \quad B_1 = 0.012186 \, 4, \quad B_2 = 0.053556 \, 80, \) and \( B_3 = 1.113451 \, 201 \), from the Schott optical glass catalog (www.us.schott.com/sgt/english/products/catalogs.html).
