Phys 506 HW1: Working with Operators

1 Problem 1

1.) Compute the eigenstates of the operator $\vec{e}_n \cdot \vec{s}$, where $\vec{e}_n = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a unit vector that points in the θ, ϕ direction (we use the physicists standard where θ is the angle from the vertical and ϕ is the polar angle in the $x-y$ plane). Write $\hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma}$ and solve the problem by diagonalizing the 2×2 matrix. (remember to normalize your final answer).

Your final answer is a 2 component spinor of the form $\binom{\alpha}{\beta}$. Use only $\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, e^{i\phi}$ and numbers in your final answer. Make sure your final answer is in the form where α, the top component of the spinor, is *real*. (Look up some trig identities if your answer looks complicated; the half angle formulas will be helpful.)

Note that if we examine

$$\left(\vec{e}_n \cdot \hat{\vec{S}}\right)^2 = \frac{\hbar^2}{4}\left(\vec{e}_n \cdot \vec{\sigma}\right)^2 = \frac{\hbar^2}{4}\vec{e}_n \cdot \vec{e}_n = \frac{\hbar^2}{4}$$

we see that the eigenvalues of $\vec{e}_n \cdot \hat{\vec{S}}$ must be $\pm\frac{\hbar}{2}$ for any direction θ, ϕ ! If you know about the Stern-Gerlach experiment, this explains why it gives the result it gives.

2 Problem 2

2.) Derive the matrices corresponding to the operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ in the $l = 1$ angular momentum representation. They satisfy

$$\left(L_i\right)_{mm'} = \hbar\langle l = 1, m|\hat{L}_i\,|l = 1, m'\rangle = \hbar\left(M_i\right)_{mm'}$$

with M a dimensionless matrix.

You should find the computation of L_z is easiest because the states $|l = 1, m\rangle$ are eigenstates of $\hat{L}_z$. You may find using the raising and lowering operators and the fact that $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ make your calculations easier. (Use the result for $\hat{L}_+|lm\rangle$ etc.) (i.e, $\hat{L}_+|10\rangle = \sqrt{2}\hbar|11\rangle$, etc.)

You should find $M_x = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \end{pmatrix}$ $\quad M_y = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$ and $M_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Note that because the $\hat{L}_z$ eigenvalues are $\hbar, 0$, and $-\hbar$, we have

$$\begin{gathered} (M_z - 1)\, M_z\, (M_z + 1) = 0 \\ \Rightarrow M_z \left(M_z^2 - 1\right) = 0 \\ \text{or } M_z^3 = M_z \end{gathered}$$

But since this is true for any direction, we have $M_i^3 = M_i$.

Indeed, just like we argued about spin $\frac{1}{2}$ above, we should have $\left(\vec{e}_n \cdot \vec{M}\right)^3 = \left(\vec{e}_n \cdot \vec{M}\right)$ with $\vec{e}_n = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$. You now will show this.

First compute

$$\vec{e}_n \cdot \vec{M} = \begin{pmatrix} \cos\alpha & \frac{1}{\sqrt{2}}\sin\alpha e^{-i\beta} & 0 \\ \frac{1}{\sqrt{2}}\sin\alpha e^{i\beta} & 0 & \frac{1}{\sqrt{2}}\sin\alpha e^{i\beta} \\ 0 & \frac{1}{\sqrt{2}}\sin\alpha e^{i\beta} & -\cos\alpha \end{pmatrix},$$

Then compute $\left(\vec{e}_n, \vec{M}\right)^2$ and $\left(\vec{e}_n \cdot \vec{M}\right)^3$ to verify

$$\left(\vec{e}_n \cdot \vec{M}\right)^3 = \left(\vec{e}_n \cdot \vec{M}\right).$$

Use this result to show that

$$\exp[i\vec{v}\cdot\vec{M}] = \sum_{n=0}^{\infty}\frac{(i)^n}{n!}|v|^n\left(\vec{e}_v\cdot\vec{M}\right)^n$$
$$= \mathbb{1} + i\sin|v|\left(\vec{e}_\sigma\cdot\vec{M}\right) + (\cos|v|-1)\mid\vec{e}_v\cdot\vec{M}\Big)^2,$$

with $\vec{e}_v = \frac{\vec{v}}{|v|} = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$.

This is another case where we can explicitly compute the exponential of a matrix. If you wish to try, it does not work for any higher angular momentum.

3 Problem 3

3.) Using what you know about exponentials of operators $e^{\hat{A}}e^{\hat{B}}$ show that, in general, we have

$$e^{i\vec{v}\cdot\vec{\sigma}}e^{i\vec{v}'\cdot\vec{\sigma}} \neq e^{i(\vec{v}+\vec{v}')\cdot\sigma}$$

Under what circumstances are they equal (this will be a relation between $\vec{v}$ and $\vec{v}'$)?

Hint: Consider BCH for Pauli matrices; do not try to multiply the matrices for $e^{i\vec{v}\cdot\vec{\sigma}}$ and $e^{i\vec{v}'\cdot\vec{\sigma}}$.

4 Problem 4

4.) Working with the $l = 1$ angular momentum matrices, compute $e^{-i\theta M_z}M_i e^{i\theta M_z}$. Use the Hadamard relation (which holds for matrices). Note that the commutators never terminate, but they do eventually repeat in a pattern. Determine what the pattern yields in terms of trig functions.

5 Problem 5

5.) Consider the symplectic group algebra

$$[K_0, K_\pm] = \pm K_\pm \quad [K_+, K_-] = -2K_0$$

This is the same as the SU(2) algebra, but there is a minus sign on the K_0 operator.

Verify that $K_+ = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ $K_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $K_0 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ satisfy the above algebra.

Compute $\exp\left(-\xi K_+ + 2i\eta K_0 + \xi^* K_-\right)$, where ξ and η are complex numbers.
Hint: First compute $\left(-\xi K_+ + 2i\eta K_0 + \xi^* K_-\right)^2$ and use that result to simplify your work. Review hyperbolic functions if the power series are unfamiliar. Your final result will have the form

$$\begin{pmatrix} K^* & \lambda^* \\ \lambda & K \end{pmatrix} \qquad (K \text{ and } \lambda \text{ are functions of } \xi \text{ and } \eta)$$

Factorize this to show the exponential disentangling identity for the symplectic group given by

$$\exp\left[-\xi K_+ + 2i\eta K_0 + \xi^* K_-\right] = e^{-\frac{\lambda}{K^*}K_+} e^{-2\ln K^* K_0} e^{\frac{\lambda^*}{K^*}K_-}.$$

6 Problem 6

6.) In lecture 2, we derived the the following simplified BCH formula

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2}[\hat{A},\hat{B}] + \frac{1}{12}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{12}[\hat{B},[\hat{B},\hat{A}]]},$$

which is exact if

$$[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] = 0$$
$$[\hat{B}, [\hat{A}, [\hat{A}, \hat{B}]]] = 0$$
$$[\hat{A}, [\hat{B}, [\hat{B}, \hat{A}]]] = 0$$
$$[\hat{B}, [\hat{B}, [\hat{B}, \hat{A}]]] = 0.$$

We want to re-express this in a different form.

Let $\hat{X} = \hat{A}$ and $\hat{Y} = \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}]$

Then

$$\begin{aligned} \hat{A} = \hat{X} \text{ and } \hat{B} &= \hat{Y} - \frac{1}{2}[\hat{A}, \hat{B}] \\ &= \hat{Y} - \frac{1}{2}\left[\hat{X}, \hat{Y} - \frac{1}{2}[\hat{A}, \hat{B}]\right] \\ &= \hat{Y} - \frac{1}{2}[\hat{X}, \hat{Y}] + \frac{1}{4}[\hat{X}, [\hat{X}, \hat{Y}]] \end{aligned}$$

since higher-order terms vanish.

Rearrange the BCH formula to its equivalent form

$$\begin{aligned} e^{\hat{X}} e^{\hat{Y}} e^{-\frac{1}{2}[\hat{X},\hat{Y}]} e^{-\frac{1}{3}[\hat{Y},[\hat{Y},\hat{X}]]} e^{\frac{1}{6}[\hat{X},[\hat{X},\hat{Y}]]} \\ = e^{\hat{X}+\hat{Y}} \end{aligned}$$

(show your work, and recall $[\hat{X}[\hat{X}, \hat{Y}]]$ and $[\hat{Y}, [\hat{Y}, \dot{X}]]$ commute with everything.)

Now consider the time evolution of a particle mowing in a linear potential with

$$\hat{H} = \frac{\hat{p}^2}{2m} + F\hat{x} \qquad \text{(a gravitational potential)}$$

The time evolution operator is $e^{-i\hat{H}t} = e^{-it[\frac{\hat{p}^2}{2m} + F\hat{x}]}$. Using the notation from earlier in the problem, pick $\hat{X} = -it\frac{\hat{p}^2}{2m}$ $\quad \hat{Y} = -itF\hat{x}$, with $[\hat{x}, \hat{p}] = i\hbar$. Use the BCH formula you derived above to compute a factorized form of $e^{-i\hat{H}t}$. Your answer will have four factors in it. Be careful. The order of the factors matters.