PHYS 5002: Homework 12

1 Creation and annihilation operator gymnastics

Consider a two level system described by fermionic creation and annihilation operators.

$$\{c^\dagger_{i\sigma}, c_{j\sigma'}\} = \delta_{ij}\delta_{\sigma\sigma'}$$
$$\{c^\dagger_{i\sigma}, c^\dagger_{j\sigma'}\} = \{c_{i\sigma}, c_{j\sigma'}\} = 0$$

for $i, j = 1, 2$ and $\sigma, \sigma' = \uparrow\downarrow$

a.) Construct the <u>total</u> spin operators $\hat{S}^z, \hat{S}^+$, and $\hat{S}^-$. Verify, using the second quantized form that

$$[\hat{S}^+, \hat{S}^-] = 2\hat{S}^z,\ [\hat{S}^z, \hat{S}^\pm] = \pm\hat{S}^\pm$$

b.) The pseudospin operators are

$$\hat{J}^+ = c^\dagger_{1\uparrow}c^\dagger_{1\downarrow} - c^\dagger_{2\uparrow}c^\dagger_{2\downarrow}$$
$$\hat{J}^- = c_{1\downarrow}c_{1\uparrow} - c_{2\downarrow}c_{2\uparrow}$$
$$\hat{J}_z = \frac{1}{2}\left(c^\dagger_{1\uparrow}c_{1\uparrow} + c^\dagger_{1\downarrow}c_{1\downarrow} + c^\dagger_{2\uparrow}c_{2\uparrow} + c^\dagger_{2\downarrow}c_{2\downarrow}\right) - 1$$

Show the pseduospin operators satisfy the $SU(2)$ algebra:

$$[\hat{J}^+, \hat{J}^-] = 2\hat{J}^z,\ [\hat{J}^z, \hat{J}^\pm] = \pm\hat{J}^\pm$$

Also, show that the spin and psuedospin operators commute. That is,

$$[\hat{J}^\pm, \hat{S}^\pm] = [\hat{J}^\pm, \hat{S}^z] = [\hat{J}^z, \hat{S}^\pm] = [\hat{J}^z, \hat{S}^z] = 0$$

c.) Find J, m_J, s, m_s for the following states:

$$c^\dagger_{1\uparrow}|0\rangle,\ c^\dagger_{1\uparrow}c^\dagger_{2\uparrow}|0\rangle,\ \frac{1}{\sqrt{2}}(c^\dagger_{1\uparrow}c^\dagger_{2\downarrow} \pm c^\dagger_{1\downarrow}c^\dagger_{2\uparrow})|0\rangle,\ \frac{1}{\sqrt{2}}(c^\dagger_{1\uparrow}c^\dagger_{1\downarrow} - c^\dagger_{2\uparrow}c^\dagger_{2\downarrow})|0\rangle$$