HW2

1 Bogliubov transformation

a.) Consider a Hamiltonian of the form

$$\hat{H} = \frac{D_1}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}\right) + \frac{D_2}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) + D_3$$

with $D_2 > D_1 > 0$. Here $\hat{a}$ and $\hat{a}^\dagger$ are simple harmonic oscillator raising and lowering operators with $\left[\hat{a}_1\hat{a}^\dagger\right] = 1$. Determine an analytic formula for the energy levels in terms of D_1, D_2, D_3 and integers.

Hint: Consider a new set of raising and lowering operators $\hat{A} = \hat{a}\cosh\theta + \hat{a}^\dagger\sinh\theta$ and $\hat{A}^\dagger = \hat{a}^\dagger\cosh\theta + \hat{a}\sinh\theta$. Verify that $\left[\hat{A}, \hat{A}^\dagger\right] = 1$ and then find a way to pick a θ such that $\hat{H} = C_1\hat{A}^\dagger\hat{A} + C_2$. From this form, you should be able to read off the spectrum.

b.) Consider our squeezed state

$$\hat{S}(\xi, \eta)|0\rangle.$$

Show that there is a linear combination

$$\hat{A} = \hat{a}\cosh\phi + \hat{a}^\dagger\sinh\phi$$

for some ϕ such that $\hat{A}$ annihilates the squeezed state:

$$\hat{A}\hat{S}(\xi, \eta)|0\rangle = 0$$

This means that we can think of the squeezed vacuum as the ground state of a Hamiltonian of the form $C_1\hat{A}^\dagger\hat{A} + C_2$. You need to find ϕ as a function of ξ and η.

Note that ϕ is generically complex and you can write the answer in terms of inverse functions.

2 Simple harmonic oscillator wavefunction in momentum space

a.) Repeat the derivation of the wavefunctions for the simple harmonic oscillator, but now in momentum space. First, verify that $|p\rangle = e^{i\frac{p\hat{x}}{\hbar}}|p{=}0\rangle$.

Second, define

$$\begin{aligned}\phi_n(p) &= (i)^n\langle p|n\rangle \\ &= (i)^n\frac{1}{\sqrt{n!}}\langle p{=}0|e^{-\frac{i}{\hbar}ip\hat{x}}\left(\hat{a}^\dagger\right)^n|0\rangle.\end{aligned}$$

Use operator methods to find the wavefunction, which looks schematically similar to (polynomial in p) times exp (polynomial in p)

b.) When computing the wavefunction in position space, we argued we needed to convert the $\hat{p}$ operator in the exponent into an $\hat{x}$ operator, so it can annihilate against $\langle x{=}0|$. We did this by breaking the $\exp\left(\frac{i}{\hbar}x\hat{p}\right)$ into an $\exp(\hat{a}^\dagger)$ and $\exp(\hat{a})$ factors, moved the $\exp(\hat{a})$ factor to the right where it annihilated against $|0\rangle$. Then introduced a new $\exp(\hat{a})$ factor and moved to the left, finally combining with $\exp(\hat{a}^\dagger)$to get $\exp(\hat{x})$. We can shorten the derivation by introducing the correct $\exp(\hat{a})$ factor on the right acting on $|0\rangle$ (multiply by one trick) move it to the left and combine with the $\exp(\hat{p})$ factor to make an $\exp(\hat{x})$ factor. Find the correct $\exp(\alpha\hat{a})$ factor to introduce on the right and show the steps needed to verify that

$$\psi_n(x) = \frac{1}{\sqrt{n!}}e^{-\frac{mw_0x^2}{2\hbar}}\langle x{=}0|\left(a^\dagger + \sqrt{\frac{2mw_0}{\hbar}}\right)^n|0\rangle$$

en route to finding the position wavefunction.

3 Displacement operator

Consider an arbitrary position translation followed by a momentum translation:

$$\exp\left(\frac{i}{\hbar}p_0\hat{x}\right)\exp\left(-\frac{i}{\hbar}x_0\hat{p}\right).$$

Combine both operators into one single exponent and replace $\hat{x}$ and $\hat{p}$ in terms of the $\hat{a}$ and $\hat{a}^\dagger$ operators to rewrite the operator as

$$D(\alpha)e^{i\phi}.$$

Determine α and ϕ in terns of x_0 and p_0. Since the overall factor $e^{i\phi}$ plays no role in wavefunctions, we can drop it when constructing the coherent state

$$D(\alpha)|0\rangle = |\alpha\rangle$$

How does the state change if we translate momentum first and then translate position?

4 Uncertainty in coherent and squeezed states

a.) When examining the general coherent states as a function of time, we found

$$\begin{aligned} e^{-i\hat{H}t}D(\alpha)|0\rangle &= e^{-i\frac{\omega_0 t}{2}}\left|\alpha e^{-i\omega_0 t}\right\rangle \\ &= e^{\frac{-i\omega_0 t}{2}}D\left(\alpha e^{-i\omega_0 t}\right)|0\rangle. \end{aligned}$$

Compute the expectation value of $\hat{x}$ and $\hat{p}$ as functions of time along with $(\Delta x)_{\alpha e^{-i\omega_0 t}}$ and $(\Delta p)_{\alpha e^{-i\omega_0 t}}$. Show that the system is always in a minimum uncertainty state. Explain how the uncertainty in position and momentum change with time. Express your results in terms of x_0 and p_0, using the α you found in problem 3.

Recall: $(\Delta\hat{O})^2\psi = \langle\psi|\hat{O}^2|\psi\rangle - \langle\psi|O|\psi\rangle^2$

b.) For the squeezed vacuum (not the displaced squeezed vacuum), we saw that

$$\begin{aligned} e^{-i\hat{H}t}\hat{S}(\xi,\eta)(0) &= e^{-i\frac{\omega_0 t}{2}}\hat{S}\left(\xi e^{-2i\omega_0 t},\eta)|0\right\rangle \\ &= \left|\xi e^{-2i\omega_0 t},\eta\right\rangle \end{aligned}$$

Pick $\xi = re^{i\phi}$ and $\eta = 0$ and determine the expectation value of $\hat{x}$ and $\hat{p}$ as functions of time along with $(\Delta x)_{\xi e^{-i\omega_0 t},\eta=0}$ and $(\Delta p)_{\xi e^{-i\omega_0 t},\eta=0}$. Show that $\Delta x\Delta p = \hbar/2$. Explain how Δx and Δp vary with time.

5 Factorization method for the particle in a box

a.) In class, we showed the original Schrödinger factorization method for a particle in an infinite square well. Schrödinger described this as "shooting sparrows with artillery". We can proceed in another fashion.

Take the potential to be zero between $-\frac{L}{2} \le x \le \frac{L}{2}$. Consider the lowering operator

$$\hat{A}_k = \frac{1}{\sqrt{2m}}\big(\hat{p} - i\hbar k\tan(k\hat{x})\big).$$

Show that $\hat{H} = \hat{A}_k^\dagger\hat{A}_k + E_k$, where you need to determine E_k.

Now, consider increasing k. Starting from $k = 0$, we see that E_k increases until k reaches $\frac{\pi}{2}$. This is the same solution we examined in class. But now, for excited states, instead of using the Sctrödinger factorization method again, lets just consider increasing k further. The E_k continues to increase, but we will find that when $\tan k\frac{L}{2} = \infty$ again, we find another excited state and so on. The idea is that we increase k until each time $\tan k\frac{L}{2}$ diverges. This condition coincides with $\psi\left(\pm\frac{L}{2}\right) = 0$.

Verify that the energies and wavefunctions are given by the well-known results for the particle in a box.

b.) Now consider a potential that is finite

$$V(x) = \begin{cases} -V_0 & |x| \leqslant \frac{L}{2} \\ 0 & |x| > \frac{L}{2} \end{cases}$$

for $|x| \leqslant \frac{L}{2}$. Use $A_k = \frac{1}{\sqrt{2}}(\hat{p} - i\hbar k \tan(k\hat{x}))$ and
for $|x| \geqslant \frac{1}{2}$ use $A_k = \frac{1}{\sqrt{2m}}(\hat{p} \pm i\hbar K)$ (decide whether $+$ or $-$ for $x < -\frac{L}{2}$ and $x > \frac{L}{2}$).
Let $\phi = \frac{kL}{2} = \sqrt{\frac{2m(V_0+E)}{\hbar^2}}\frac{L}{2}$, $\quad \kappa = \sqrt{-\frac{2mE}{\hbar^2}}$ and $\phi_0 = \sqrt{\frac{2mV_0}{\hbar^2}}\frac{L}{2}$ (recall E<0 for bound states). Use the requirement that $\hat{A}_k$ is continuous at $x = \pm\frac{L}{2}$, to find a transcendental equation that determines a valid solution (this requirement comes from conservation of probability current). Note that any k value that satisfies this equation yields a valid solution.

Determine the wavefunctions (unnormalized).

This yields all of the even solutions. One can also find the odd ones but I won't ask you to.