PHYS 5002: Homework 5

1 A Perturbed Hydrogen Atom

Consider the Hamiltonian:

$$\hat{H} = \frac{\hat{p}_r^2}{2\mu} + \frac{\hat{L}^2}{2\mu\hat{r}^2} - \frac{e^2}{\hat{r}} + \frac{\hbar^2\gamma}{2\mu\hat{r}^2}$$

where γ is a dimensionless constant.

Use the factorization method to compute all of the energy eigenvalues. Introduce an appropriate principal quantum number.

You may find it helpful to define a parameter λ that satisfies

$$\lambda(\lambda+1) = l(l+1) + \gamma$$

Further note that as $\gamma \to 0$, you should recover the energy levels of hydrogen. For $\gamma \neq 0$, you should find that the energy levels depend on both n and l now.

Use the factorization method as described in the previous HW. For each l define an auxiliary Hamiltonian for different m's and construct eigenstates from $\hat{B}^\dagger$'s acting on the ground state. Note that you cannot use the shortcut method we had used before.

ANSWER:

$$E_{n,l} = -\frac{e^2}{2a_0} \frac{1}{n^2 + \gamma + 2(n - l - \frac{1}{2})\left(\sqrt{(l+\frac{1}{2})^2 + \gamma} - (l + \frac{1}{2})\right)}$$

2 Clebsch-Gordan Coefficients

Find the Clebsch-Gordan coefficients for combining an arbitrary angular momentum j_1 with $j_2 = 1$ and $m_2 = +1, 0, -1$.

That is, find the nine entries in the 3×3 table

$$\langle j_1, m_1 = m_j - m_2, j_2 = 1, m_2 | j, m_j, j_1, j_2 = 1\rangle$$

	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$j = j_1 + 1$			
$j = j_1$			
$j = j_1 - 1$			

3 The LMG Model

The following model was studied by Lipkin, Meshkov, and Glick and is sometimes called the Lipkin model or the LMG model. It is also just an Ising model in a transverse field with all-to-all coupling.

$$\hat{H} = -2\sum_{i=1}^{n}\sum_{j>i} \hat{S}_i^z \hat{S}_j^z + B\sum_{i}^{N} \hat{S}_i^x$$

where each $\hat{S}_i$ is spin one-half.

a.) Show we can rewrite the Hamiltonian as

$$\hat{H} = -(\hat{S}^z_{tot})^2 + B\hat{S}^x_{tot} + \text{const.}$$

by finding the constant. Note that $\hat{S}^\alpha_{tot} = \sum_{i=1}^N \hat{S}^\alpha_i$ is the total spin in the α-direction. Recall what the square of a Pauli matrix is to solve this problem quickly.

b.) Using the form found in (a), show that $\hat{H}$ commutes with the total angular momentum squared.

$$[\hat{H}, (\hat{S}^x_{tot})^2 + (\hat{S}^y_{tot})^2 + (\hat{S}^z_{tot})^2] = 0.$$

This means that states can be classified according to their total S values:

$$\hat{S}^2_{tot} \left|\text{state}\right\rangle = \hbar^2 S(S+1) \left|\text{state}\right\rangle .$$

But, because $[\hat{H}, \hat{S}^z_{tot}] \neq 0$ the different m_s states are coupled by $\hat{H}$ hence $\left\langle S, m_s \middle| \hat{H} \middle| S, m_s' \right\rangle$ is not proportional to $\delta_{m_s,m_s'}$

c.) Consider $N = 4$. There are $2^4 = 16$ total states. They decompose into an $S = 2$, three $S = 1$, and two $S = 0$ states. It is best to construct them starting from spin one-half and adding spins up to $N = 4$ to see the different states.

Compute $(H_S)_{mm'} = \left\langle S, m \middle| \hat{H} \middle| S, m \right\rangle$ for $S = 2, 1, 0$.

Note the matrix is the same for each S value and so there are only three matrices to compute. You do not need to generate all the states explicitly. Use the facts that $\hat{S}^x = \frac{\hat{S}^+ + \hat{S}^-}{2}$ and angular momentum commutation relations.

d.) Set $\hbar = 1$ and choose dimensionless units for B. Plot the eigenvalues $E(B)$ versus B for $0 \leq B \leq 4$. This entails using three different colors (one for each S).