PHYS 5002: Homework 9

1 Perturbation of a driven system

A system with degenerate eigenstates that is subjected to a time-independent perturbation is being driven at its resonance frequency ($\omega = 0$). Consider the following example.

A Hydrogen atom prepared in its ground state is prepared for $t \leq 0$ with spin-up along the z-axis. At time $t = 0$, a constant **B**-field is turned on which points in an arbitrary direction (θ, ϕ). Neglect fine structure (and $\mathbf{A}^2$ terms). Compute the probability that the atom will be found in the ground state with spin-down as a function of time.

Do the problem first with first-order perturbation theory and then solve it exactly. Discuss the accuracy of the perturbation theory.

This type of problem is called a quench problem, because we have suddenly changed the Hamiltonian at $t = 0$ to another Hamiltonian.

2 Nailed down spin-half particle

A "nailed-down" spin-$\frac{1}{2}$ particle is acted on by a constant magnetic field in the z-direction and by an oscillatory field in the xy-plane.

$$\hat{H} = \hat{H}_0 + \hat{V}(t),\ \hat{H}_0 = \hbar\Omega_0\hat{S}_z,\ \hat{V}(t) = \hbar\Omega_1(\hat{S}_x \cos\omega t + \hat{S}_y \sin\omega t)$$

a.) At $t = 0$ the particle is in the spin-up state along the z-axis. What is the probability that it will be found up at time t? (Solve the problem exactly)

b.) Use time-dependent perturbation theory to second-order to compute the probability.

c.) Compare the perturbation theory to the exact result expanded to second-order. Comment on the accuracy of the perturbation theory.

3 Time-ordered product gymnastics

Consider the time-dependent harmonic oscillator:

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$$

where

$$\hat{H}_0 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right),\ \hat{V}(t) = C\left(e^{i\Omega t}\hat{a}^\dagger + e^{-i\Omega t}\hat{a}\right)$$

a.) Compute $\hat{U}_S(t,0)$ from the interaction representation formula

$$\hat{U}_S(t,0) = e^{-\frac{i}{\hbar}\hat{H}_0 t} T e^{-\frac{i}{\hbar}\int_0^t dt' \hat{V}_I(t')}$$

to second order in $\hat{V}$.

b.) Compute $\hat{U}_S(t,0)$ from the formal solution of Schrödinger's equation

$$\hat{U}_S(t,0) = T e^{-\frac{i}{\hbar}\int_0^t dt' \hat{H}(t')}$$

to second order in $\hat{H}$.

c.) Compare the two results and comment on the differences.