Phys 506 lecture 1: Spin and Pauli matrices

This lecture should be primarily a review for you of properties of spin one-half. I do suspect that some of the identities derived here, especially the exponential disentangling identity, will be new for you.

1 Spin operators and states

Recall the spin operators and eigenstates. The states $|\uparrow; z\rangle$ and $|\downarrow; z\rangle$ are eigenstates of $\hat{S}_z$, which satisfy

$$\left.\begin{array}{l}\hat{S}_z|\uparrow; z\rangle = \frac{\hbar}{2}|q; z\rangle \\ \hat{S}_z|\downarrow; z\rangle = -\frac{\hbar}{2}|\downarrow; z\rangle\end{array}\right\} \text{ eigenstates.}$$

Next, we discuss the raising and lowering operators. Define $\hat{S}_+$and $\hat{S}_-$to connect these states

$$\hat{S}_+\,|\uparrow; z) = 0, \quad \hat{S}_+|\downarrow; z\rangle = \hbar|\uparrow; z\rangle$$
$$\hat{S}_-|\uparrow; z\rangle = \hbar|\downarrow; z\rangle, \quad \hat{S}_-\,|\downarrow; z\rangle = 0.$$

Compute their commutators by acting the operator each, in turn, onto the states. Note that the commutator is computed via the action on the states themselves, using the above rule. We cannot determine them any other way. We have

$$\left(\hat{S}_+\hat{S}_- - \hat{S}_-\hat{S}_+\right)|\uparrow; z\rangle = \hbar^2|\uparrow; z\rangle = 2\hbar\hat{S}_z|\uparrow; z\rangle$$
$$\left(\hat{S}_+\hat{S}_- - \hat{S}_-\hat{S}_+\right)|\downarrow; z\rangle = -\hbar^2|\downarrow; z\rangle = 2\hbar\hat{S}_z|\downarrow; z\rangle$$

So $\boxed{\left[\hat{S}_+, \hat{S}_-\right] = 2\hbar\hat{S}_z}$

Similarly:

$$\left(\hat{S}_z\hat{S}_+ - \hat{S}_+\hat{S}_z\right)|\uparrow; z\rangle = 0 = \hat{S}_+|\uparrow; z\rangle$$
$$\left(\hat{S}_z\hat{S}_+ - \hat{S} + \hat{S}_z\right)|\downarrow; z\rangle = \hbar^2|\downarrow; z\rangle = \hbar\hat{S}_+|\downarrow; z\rangle$$

So $\boxed{\left[\hat{S}_z, \hat{S}_+\right] = \hbar\hat{S}_+}$. You can verify yourself that $\boxed{\left[\hat{S}_z, \hat{S}_-\right] = -\hbar\hat{S}_-}$.

Please ensure that you are comfortable calculating the commutators in this way, by acting the operators onto the states in the order provided and determining what the final result is. Because we acted them on all the states in our Hilbert space, we can use this to determine the commutation rule for the operators themselves. This is how we obtained our summary equations.

These three commutation relations are the $SU(2)$ algebra.

2 Cartesian spin operators

Next, we move on to determine the Cartesian spin operators. If we define $\hat{S}_x = \frac{1}{2}\left(\hat{S}_+ + \hat{S}_-\right)$ and $\hat{S}_y = \frac{1}{2i}\left(\hat{S}_+ - \hat{S}_-\right)$, then $\hat{S}_+ = \hat{S}_x + i\hat{S}_y$ and $\hat{S}_- = \hat{S}_x - i\hat{S}_y$. The algebra of the Cartesian spins follows:

$$\begin{aligned}\left[\hat{S}_x, \hat{S}_y\right] &= \frac{1}{4i}\left[\hat{S}_+ + \hat{S}_-, \hat{S}_+ - \hat{S}_-\right] \\ &= \frac{1}{4i}\left[\left[\hat{S}_+, \hat{S}_+\right] - \left[\hat{S}_+, \hat{S}_-\right] + \left[\hat{S}_-, \hat{S}_+\right] - \left[\hat{S}_x, \hat{S}\right]\right\} \\ &= \frac{1}{4i}\left[-2\hbar\hat{S}_z - 2\hbar\hat{S}_z\right] = i\hbar\hat{S}_z\end{aligned}$$

where we used the commutation relations we already knew to evaluate them.

You can verify (and should) that we have

$$\left[\hat{S}_i, \hat{S}_j\right] = i\hbar\sum_k \varepsilon_{ijk}\hat{S}_k$$

which is the standard form for angular momentum commutators. Here, $\varepsilon_{\mathrm{ijk}}$ is the completely antisymmetric tensor, which satisfies $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$ and $\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$, and all others vanish. Note we freely use 1, 2, 3 or x, y, z. It should be clear from the context what we mean.

From this relation, we can compute the commutation relation of the total spin squared with the Cartesian angular momentum operators. We have

$$\left[\hat{S}^2, \hat{S}_j\right] = 0.$$

Proof: $\hat{S}^2 = \sum_i \hat{S}_i\hat{S}_i$ so

$$\begin{aligned}\left[\hat{S}^2, \hat{S}_j\right] &= \sum_i\left[\hat{S}_i\hat{S}_i, \hat{S}_j\right] = \sum_i\left(\hat{S}_i\left[\hat{S}_i, \hat{S}_j\right] + \left[\hat{S}_i, \hat{S}_j\right]\hat{S}_i\right) \\ &= i\hbar\sum_i\sum_k\left(\hat{S}_i\varepsilon_{ijk}\hat{S}_k + \varepsilon_{ijk}\hat{S}_k\hat{S}_i\right) \\ &= i\hbar\sum_{ik}\varepsilon_{ijk}\left(\hat{S}_i\hat{S}_k + \hat{S}_k\hat{S}_i\right).\end{aligned}$$

The claim is that this is zero. To see this let $i \to k'$ and $k \to i'$

$$\begin{aligned}&= i\hbar\sum_{i'k'}\varepsilon_{k'ji'}\left(\hat{S}_{k'}\hat{S}_{i'} + \hat{S}_{i'}\hat{S}_{k'}\right) \\ &= i\hbar\sum_{i'k'}(-\varepsilon_{i'jk'})\left(\hat{S}_{i'}\hat{S}_{k'} + \hat{S}_{k'}\hat{S}_{i'}\right)\end{aligned}$$

Since $\varepsilon_{ijk} = -\varepsilon_{kji}$. Now drop the primes

$$= -i\hbar \sum_{ik} \varepsilon_{ijk} \left(\hat{S}_i \hat{S}_k + \hat{S}_k \hat{S}_i \right)$$

Anything equal to its negative must vanish, so $\left[\hat{S}^2, \hat{S}_j \right] = 0$.

Let's compute $\hat{S}^2 |\sigma; z\rangle$ with $\sigma =\uparrow$ or $\downarrow$. First, we express the square of the spin in the spherical basis, because we know what those operators do when acting on states:

$$\begin{aligned} \hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 &= \frac{1}{4} \left(\hat{S}_+ + \hat{S}_- \right)^2 - \frac{1}{4} \left(\hat{S}_z - \hat{S}_- \right)^2 + \hat{S}_z^2 \\ &= \frac{1}{2} \left(\hat{S}_+ \hat{S}_- + \hat{S}_- \hat{S}_+ \right) + \hat{S}_z^2 \end{aligned}$$

Then we evaluate directly on states. So, acting on the up spin gives

$$\begin{aligned} \hat{S}^2 | \uparrow; z\rangle &= \frac{1}{2} \left(\hat{S}_+ \hat{S}_- + \hat{S}_- \hat{S}_+ \right) | \uparrow; z\rangle + \hat{S}_z^2 | \uparrow; z\rangle \\ &= \left(\frac{1}{2} \hbar^2 + \frac{1}{4} \hbar^2 \right) | \uparrow; z\rangle = \frac{3}{4} \hbar^2 | \uparrow; z\rangle \end{aligned}$$

And acting on the down spin gives

$$\begin{aligned} \hat{S}^2 (\downarrow; z) &= \frac{1}{2} \left(\hat{S}_+ \hat{S}_- + \hat{S}_- \hat{S}_+ \right) | \downarrow; z\rangle + \hat{S}_z^2 | \downarrow; z\rangle \\ &= \frac{3}{4} \hbar^2 (\downarrow; z) \end{aligned}$$

as well. Hence, $|\sigma; z\rangle$ is an eigenstate of both $\hat{S}^2$ and $\hat{S}_z$!

3 Matrix representation of spin

We next look at the matrix representations of spin. Assume we have an arbitrary superposition of states

$$|\psi\rangle = \alpha | \uparrow; z\rangle + \beta | \downarrow; z\rangle$$

We let the column vector $\binom{\alpha}{\beta}$ denote the state $|\psi\rangle$. The operators $\hat{S}_i$ are represented by two-dimensional matrices in this space. For example

$$\begin{array}{c} \hat{S}_z \mid \hat{\uparrow}; z\rangle = \frac{\hbar}{2} | \uparrow; z\rangle \\ \hat{S}_z \mid \downarrow; z\rangle = \frac{\hbar}{2} | \downarrow; z\rangle \end{array} \Leftrightarrow \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The matrix representing the operator is

$$M_{\sigma\sigma'} = \langle \sigma; z | \, \hat{S}_i \, | \sigma'; z\rangle$$

which includes 4 numbers for the matrix from the different choices of σ and σ'.

Let's compute the matrix for $\hat{S}_y$

$$S^y_{\sigma\sigma'} = \langle\sigma; z|\hat{S}_y|\sigma'; z\rangle = \frac{1}{2i}\langle\sigma; z|\hat{S}_+ - \hat{S}_-|\sigma'; z\rangle$$
$$\sigma' =\uparrow: \left(\hat{S}_+ - \hat{S}_-\right)|\uparrow; z\rangle = -\hbar\,|\downarrow; z\rangle \Rightarrow S^y_{\uparrow\uparrow} = 0, \quad S^y_{\downarrow\uparrow} = \frac{i\hbar}{2}$$
$$\sigma' =\downarrow: \left(\hat{S}_+ - \hat{S}_-\right)|\downarrow; z\rangle = \hbar|\uparrow; z\rangle \Rightarrow S^y_{\uparrow\downarrow} = -\frac{i\hbar}{2}, \ \hat{S}^y_{\downarrow\downarrow} = 0.$$

So $\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Similarly, we have $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We call σ_i the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that $\sigma_i^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow S^2_{\sigma\sigma'} = \frac{\hbar^2}{4} \times 3 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

These matrices also anticommute. To see this go back to our spin commutators

$$\hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x = i\hbar\hat{S}_z.$$

Multiply on left by $\hat{S}_x$ and on right by $\hat{S}_x$

$$\hat{S}_x^2\hat{S}_y - \hat{S}_x\hat{S}_y\hat{S}_x = i\hbar\hat{S}_x\hat{S}_z$$
$$\hat{S}_x\hat{S}_y\hat{S}_x - \hat{S}_y\hat{S}_x^2 = i\hbar\hat{S}_z\hat{S}_x.$$

Now substitute in the Pauli matrices

$$\frac{\hbar^3}{8}\left[\sigma_x^2\sigma_y - \sigma_x\sigma_y\sigma_x\right] = i\frac{\hbar^3}{4}\sigma_x\sigma_z$$
$$\frac{\hbar^3}{8}\left[\sigma_x\sigma_y\sigma_x - \sigma_y\sigma_x^2\right] = i\frac{i\hbar^3}{4}\sigma_z\sigma_x.$$

Add (and recall $\sigma_x^2 = \mathbb{1}$):

$$\frac{\hbar^3}{8}\left[\sigma_y - \sigma_y\right] = i\frac{\hbar^3}{4}\left(\sigma_x\sigma_z + \sigma_z\sigma_x\right)$$
$$0 = \sigma_x\sigma_z + \sigma_z\sigma_x$$

and it is obvious this holds for other permutations too.

Hence, we have derived that

$$\sigma_i\sigma_j = \frac{1}{2}\left[\sigma_i\sigma_j + \sigma_j\sigma_i + \sigma_i\sigma_j - \sigma_j\sigma_i\right]$$
$$= \frac{1}{2}\delta_{ij} \times 2\mathbb{1} + \frac{1}{2}i2\varepsilon_{ijk}\sigma_k$$
$$\text{or } \sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k$$

This last equation is an important relation to remember about the product of two Pauli spin matrices.

Another interesting identity is the product of all 3

$$\sigma_x\sigma_y\sigma_z = i\varepsilon_{123}\sigma_z\sigma_z = i\mathbb{1}$$
$$\sigma_x\sigma_y\sigma_z = i\mathbb{1}.$$

Any 2×2 matrix can be expressed in terms of the identity and the 3 Pauli matrices. This is called completeness over the space of 2×2 matrices.
Check: $\quad \alpha\mathbb{1} + \beta\sigma_x + \gamma\sigma_y + \delta\sigma_z = \begin{pmatrix} \alpha+\delta & \beta-i\gamma \\ \beta+i\gamma & \alpha-\delta \end{pmatrix}$
So to find $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$, we set $a=\alpha+\delta \quad b=\beta-i\gamma \; c=\beta+i\gamma d=\alpha\delta$
or $\alpha=\frac{a+d}{2} \quad \beta=\frac{b+c}{2} \quad \gamma=\frac{i}{2}(b-c)$ and $\delta=\frac{a-d}{2}$.

We often write this as $M=\alpha\mathbb{1}+\vec{v}\cdot\vec{\sigma} \quad \vec{v}=\left(\frac{b+c}{2},\frac{i}{2}(b-c),\frac{a-d}{2}\right)$.

4 Working with spin matrices

Let's get some practice working with these objects

$$\begin{aligned}
(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) &= \sum_{ij} A_iB_j\sigma_i\sigma_j \\
&= \sum_{ij} A_iB_j\left(\delta_{ij}\mathbb{1}+i\varepsilon_{ijk}\sigma_k\right) \\
&= \vec{A}\cdot\vec{B}\mathbb{1} + i\varepsilon_{ijk}A_iB_j\sigma_k \\
&= \vec{A}\cdot\vec{B}\mathbb{1} + i(\vec{A}\times\vec{B})\cdot\vec{\sigma}
\end{aligned}$$

Note that if $\vec{A}$ is parallel to $\vec{B}$, then $(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma})=\vec{A}\cdot\vec{B}\mathbb{1}$. This identity is a useful one to remember.

Next, we evaluate the matrix exponential:

$$\exp[i\vec{v}\cdot\vec{\sigma}] = \sum_{n=0}^{\infty}\frac{(i)^n}{n!}(\vec{v}\cdot\vec{\sigma})^n.$$

First separate out into even and odd powers

$$e^{i\vec{v}\cdot\vec{\sigma}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\vec{v}\cdot\vec{\sigma})^{2n} + i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(\vec{v}\cdot\vec{\sigma})^{2n+1}.$$

But recall $(\vec{v}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma})=v^2\mathbb{1}$ so

$$e^{i\vec{v}\cdot\vec{\sigma}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(v^2\right)^n\mathbb{1} + i\vec{v}\cdot\vec{\sigma}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(v^2\right)^n$$

$$\boxed{e^{i\vec{v}\cdot\vec{\sigma}} = \cos|v|\mathbb{1} + i\frac{\vec{v}\cdot\vec{\sigma}}{|v|}\sin|v|}$$

This is called the generalized Euler identity.

Let's compute the similarity transformation of a Pauli matrix (corresponding to a rotation)

$$e^{i\vec{v}\cdot\vec{\sigma}}\sigma_j e^{-i\vec{v}\cdot\sigma}.$$

In general, such terms can involve an infinite series, as we will show in a later lecture via the Hadamard lemma, but in this case, we can explicitly calculate it. Just use our generalized Euler identity for each exponential, followed by our product rule for pairs of Pauli matrices:

$$\begin{aligned}
e^{i\bar{v}\cdot\vec{\sigma}}\sigma_j e^{-i\vec{v}\cdot\vec{\sigma}} &= \left(\cos|v|\mathbb{1} + i\frac{\vec{v}\cdot\vec{\sigma}}{|v|}\sin|v|\right)\sigma_j\left(\cos|\sigma|\mathbb{1} - i\frac{\vec{v}\cdot\vec{\sigma}}{|v|}\sin(v)\right)\\
&= \cos^2|v|\sigma_j - i\frac{\cos|v|\sin|v|}{|v|}\left(\sigma_j\vec{v}\cdot\vec{\sigma} - \vec{v}\cdot\vec{\sigma}\sigma_j\right) + \frac{\sin^2|v|}{|v|^2}\vec{v}\cdot\vec{\sigma}\sigma_j\vec{v}\cdot\vec{\sigma}\\
&= \cos^2|v|\sigma_j - i\frac{\cos|v|\sin|v|}{|v|}\sum_i\left[\sigma_j,\sigma_i\right]v_i + \frac{\sin^2|v|}{|v|^2}\sum_{ik}v_iv_k\sigma_i\sigma_j\sigma_k\\
&= \cos^2|v|\sigma_j + \frac{\cos|v|\sin|v|}{v}\sum_{ik}2\varepsilon_{jik}v_i\sigma_k + \frac{\sin^2|v|}{|\sigma|^2}\sum_{ik}v_iv_k\sigma_i\sigma_j\sigma_k
\end{aligned}$$

But $\sum_{ik} v_iv_k\sigma_i\sigma_j\sigma_k = \sum_{ik} v_iv_k\left[\delta_{ij}\mathbb{1} + i\varepsilon_{ijl}\sigma_l\right]\sigma_k$. So, we have that

$$\begin{aligned}
\sum_{ik} v_iv_k\sigma_i\sigma_j\sigma_k &= v_j\vec{v}\cdot\vec{\sigma} + i\sum_{ikl}v_iv_k\varepsilon_{ijl}\left(\cancelto{0}{\delta_{lk}} + i\varepsilon_{lkm}\sigma_m\right)\\
&= v_j\vec{v}\cdot\vec{\sigma} - \sum_{iklm}v_iv_k\varepsilon_{ijl}\varepsilon_{lkm}\sigma_m\\
&= v_j\vec{v}\cdot\vec{\sigma} - \sum_{ikm}v_iv_k\left(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}\right)\sigma_m\\
&= v_j\vec{v}\cdot\vec{\sigma} - v^2\sigma_j + \vec{v}\cdot\vec{\sigma}v_j
\end{aligned}$$

So that

$$e^{i\bar{v}\cdot\vec{\sigma}}\sigma_j e^{-i\vec{v}\cdot\vec{\sigma}} = \cos^2|v|\delta_j + 2\frac{\cos|v|\sin|v|}{|v|}(\vec{v}\times\vec{\sigma})_j + \frac{\sin^2(v)}{|v|^2}\left(2\vec{v}\cdot\vec{\sigma}v_j - v^2\sigma_j\right).$$

5 Exponential disentangling

Our last topic is on exponential disentangling. This is an identity most have not seen before. It is derived by factorizing the exponential of a combination of Pauli matrices into a product of three special Pauli matrices. First note that $\sigma_+ = \sigma_x + i\delta_y = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ so

$$
\begin{aligned}
\exp(\alpha\sigma_+) &= 1 + \alpha\sigma_+ + \frac{1}{2}\alpha^2\left(\cancelto{0}{\sigma_+^2}\right) + \cdots \\
&= \begin{pmatrix} 1 & 2\alpha \\ 0 & 1 \end{pmatrix} \\
\exp(\alpha\sigma_-) &= \begin{pmatrix} 1 & 0 \\ 2\alpha & 1 \end{pmatrix}
\end{aligned}
$$

That is, the exponential of raising or lowering Paulis is given by a sum of just two terms. Now, let's consider the following exponential

$$
\exp(\vec{v}\cdot\vec{\sigma}) = \begin{pmatrix} \cos|\sigma| + i\frac{v_z}{|\sigma|}\sin|v| & \left(i\frac{v_x}{|v|} + \frac{v_y}{|v|}\right)\sin|v| \\ \left(\frac{iv_x}{|v|} - \frac{v_y}{|v|}\right)\sin|v| & \cos|v| - i\frac{v_z}{|v|}\sin|v| \end{pmatrix}
$$

We want to re-write it as $\exp[\alpha\sigma_+]\exp[\beta\sigma_z]\exp[\gamma\sigma_-]$, so we have

$$
\begin{aligned}
&\begin{pmatrix} 1 & 2\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2\gamma & 1 \end{pmatrix} \\
=&\begin{pmatrix} e^{\beta} & 2\alpha e^{-\beta} \\ 0 & e^{-\beta} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2\gamma & 1 \end{pmatrix} = \begin{pmatrix} e^{\beta} + 4\alpha\gamma e^{-\beta} & 2\alpha e^{-\beta} \\ 2\gamma e^{-\beta} & e^{-\beta} \end{pmatrix}.
\end{aligned}
$$

Comparing to the exponential, we have

$$
\begin{aligned}
\cos|v| + i\frac{v_z}{|v|}\sin|v| &= e^{\beta} + 4\alpha\gamma e^{-\beta} \\
\left(i\frac{v_x}{|\sigma|} + \frac{v_y}{|v|}\right)\sin|v| &= 2\alpha e^{-\beta} \\
\left(i\frac{v_x}{|v|} - \frac{v_y}{|v|}\right)\sin|v| &= 2\gamma e^{-\beta} \\
\cos|v| - i\frac{v_z}{|v|}\sin|v| &= e^{-\beta},
\end{aligned}
$$

First note that $e^{\beta}\left(1 + 4\alpha\gamma e^{-2\beta}\right) = \frac{1}{\cos|v| - i\frac{v_z}{|v|}\sin|v|}\left(1 + \sin^2|v|(-\frac{v_z^2 - v_y^2}{|v|^2}\right))$

$$
\begin{aligned}
&= \frac{\cos|v| + i\frac{v_z}{|v|}\sin|v|}{\cos^2|v| + \frac{v_z^2}{|v|^2\sin^2|v|}}\left(1 + \sin^2|v|\left(\frac{v_z^2}{|v|^2} - 1\right)\right) \\
&= \cos|v| + i\frac{v_z}{|v|}\sin|v|
\end{aligned}
$$

So if we solve the bottom three equations, the top automatically holds! This then gives us

$$\boxed{\beta = -\ln\left[\cos|v| - i\frac{v_z}{|v|}\sin|v|\right]} \tag{1}$$

$$\boxed{\alpha = \frac{1}{2}\left(\frac{iv_x}{|v|} + \frac{v_y}{|v|}\right)\frac{\sin|v|}{\cos|v| - \frac{iv_z}{|v|}\sin|v|}} \tag{2}$$

$$\boxed{\gamma = \frac{1}{2}\left(\frac{iv_x}{|v|} - \frac{v_y}{|\ v'}\right)\frac{\sin|v|}{\cos|v| - i\frac{v_z}{|v|}\sin|v|}} \tag{3}$$

This identity is called the exponential disentangling identity and it is an important one.

An important special case is when $\boxed{v_x = v_z = 0}$, then

$$\boxed{\beta = \ln\sec v_y \quad \alpha = \frac{1}{2}\tan v_y \quad \gamma = -\frac{1}{2}\tan v_y}$$

This is your first taste of exponential disentangling. We will see more of it soon.

What is amazing about exponential disentangling is that is holds for any angular momentum, not just spin one half. But, we will have more to say about that later.