Phys 506 Lecture 12: Addition of angular momentum

1 Combining independent angular momenta

Suppose we have two angular momenta which are independent and commute with each other, $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$. Then we have the following commutation relations:

$$\begin{aligned}[\hat{J}_{1i}, \hat{J}_{1j}] &= i\hbar\epsilon_{ijk}\hat{J}_{1k} \\ [\hat{J}_{2i}, \hat{J}_{2j}] &= i\hbar\epsilon_{ijk}\hat{J}_{2k} \\ [\hat{J}_{1i}, \hat{J}_{2j}] &= 0.\end{aligned}$$

We can then define the **total angular momentum operator** as:

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2.$$

Then:

$$[\hat{J}_i, \hat{J}_j] = [\hat{J}_{1i} + \hat{J}_{2i}, \hat{J}_{1j} + \hat{J}_{2j}] = i\hbar\epsilon_{ijk}(\hat{J}_{1k} + \hat{J}_{2k}) = i\hbar\epsilon_{ijk}\hat{J}_k.$$

We now claim put forth the following claim.

2 Uncoupled basis

Theorem 2.1. *The following operators all commute:* $\hat{J}_1^2, \hat{J}_{1z}, \hat{J}_2^2, \hat{J}_{2z}$.

Proof. This is obvious since $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ commute, and $[\hat{J}_1^2, \hat{J}_{1z}^2] = [\hat{J}_2^2, \hat{J}_{2z}] = 0$. □

As a result, we can form eigenstates with the labels $|j_1, m_1, j_2, m_2\rangle$ such that:

$$\begin{aligned}\hat{J}_1^2|j_1, j_2, m_1, m_2\rangle &= \hbar^2 j_1(j_1+1)|j_1, j_2, m_1, m_2\rangle,\\ \hat{J}_2^2|j_1, j_2, m_1, m_2\rangle &= \hbar^2 j_2(j_2+1)|j_1, j_2, m_1, m_2\rangle,\\ \hat{J}_{1z}|j_1, j_2, m_1, m_2\rangle &= \hbar m_1|j_1, j_2, m_1, m_2\rangle,\\ \hat{J}_{2z}|j_1, j_2, m_1, m_2\rangle &= \hbar m_2|j_1, j_2, m_1, m_2\rangle.\end{aligned}$$

These states are formed from the tensor product of the states with different angular momentum. The corresponding operators act on the corresponding states. As an example, we would have $|j_1, m_1\rangle \otimes |j_2, m_2\rangle = |j_1, j_2, m_1, m_2\rangle$, with the $\hat{J}_1$ operators acting on the first state in the tensor product and the $\hat{J}_2$ operators acting on the second state. Keeping the tensor product notation around is unwieldy so we drop it. This first basis is called the uncoupled basis, because the angular momenta are and remain indeendent.

3 Coupled basis

Theorem 3.1. *The following operators also commute:* $\hat{J}^2, \hat{J}_z, \hat{J}_1^2, \hat{J}_2^2$

Proof. We already know:

$$[\hat{J}_z, \hat{J}_1^2] = 0, \quad [\hat{J}_z, \hat{J}_2^2] = 0, \quad [\hat{J}^2, \hat{J}_z] = 0.$$

Hence, we need to check $[\hat{J}^2, \hat{J}_1^2] = 0$. But we can express $\hat{J}^2$ as:

$$\hat{J}^2 = \hat{J}_1^2 + 2\hat{\mathbf{J}}_1 \cdot \hat{\mathbf{J}}_2 + \hat{J}_2^2 = \hat{J}_1^2 + 2\hat{\mathbf{J}}_{1j} \cdot \hat{\mathbf{J}}_{2j} + \hat{J}_2^2$$

But $[\hat{J}_1^2, \hat{J}_{1j}] = 0$ which implies $[\hat{J}^2, \hat{J}_1^2] = 0$. Similarly $[\hat{J}^2, \hat{J}_2^2] = 0$. □

As a result, we can also label states by $|j, m_j, j_1, j_2\rangle$ such that:

$$\begin{aligned}\hat{J}^2|j, m_j, j_1, j_2\rangle &= \hbar^2 j(j+1)|j, m_j, j_1, j_2\rangle\\ \hat{J}_z|j, m_j, j_1, j_2\rangle &= \hbar m_j|j, m_j, j_1, j_2\rangle\\ \hat{J}_1^2|j, m_j, j_1, j_2\rangle &= \hbar^2 j_1(j_1+1)|j, m_j, j_1, j_2\rangle\\ \hat{J}_2^2|j, m_j, j_1, j_2\rangle &= \hbar^2 j_2(j_2+1)|j, m_j, j_1, j_2\rangle\end{aligned}$$

Both representations are completely equivalent and span the whole basis of states. This however raises the question of how do we convert between them. In addition, we might also be interested in what the allowed values are of each label, e.g. given j_1 and j_2, what values are allowed for j and m. Hence, we will focus on answering the following two questions.

4 Relationship between the two bases

Questions.

1. Given j_1 and j_2, what values of j are allowed?
2. How do I convert between the different bases? The coefficients of the expansions are called **Clebsch-Gordan coefficients**.

We will focus first on question (1). Suppose we start with two states, one with j_1 and one with j_2, and these values are fixed. Look at the representation $|j_1, m_1, j_2, m_2\rangle$. We have

$$\hat{J}_z|j_1, m_1, j_2, m_2\rangle = (\hat{J}_{1z} + \hat{J}_{2z})|j_1, m_1, j_2, m_2\rangle = \hbar(m_1 + m_2)|j_1, m_1, j_2, m_2\rangle.$$

This implies that this representation is an eigenfunction of $\hat{J}_z$ with eigenvalue $m_j = m_1+m_2$. In general, this state is not an eigenstate of $\hat{J}^2$. Examine the maximal spin state

$$|j_1, m_1 = j_1, j_2, m_2 = j_2\rangle.$$

Then $m_j = j_1+j_2$ is the maximal value for the z-component of total angular momentum. So the maximal j we can have is $j = j_1 + j_2$. In other words, this state is also

$$|j = j_1 + j_2, m_j = j_1 + j_2, j_1, j_2\rangle$$

up to a phase. Similarly, the state $|j_1, m_1 = -j_1, j_2, m_2 = -j_2\rangle$ is $|j = j_1 + j_2, m_j = -(j_1 + j_2), j_1, j_2\rangle$ up to a phase.

How do we find the state with $j = j_1 + j_2 - 1$? Look at the states with $m_j = j_1 + j_2 - 1$:

$$|j_1, m_1 = j_1 - 1, j_2, m_2 = j_2\rangle \quad \text{and} \quad |j_1, m_1 = j_1, j_2, m_2 = j_2 - 1\rangle.$$

These states must have one linear combination which has $j = j_1 + j_2$ and $m_j = j_1 + j_2 - 1$, and one which has $j = j_1 + j_2 - 1$ and $m_j = j_1 + j_2 - 1$. How do we find them however? Recall that:

$$\hat{J}_-|j, m_j, j_1, j_2\rangle \propto |j, m_j - 1, j_1, j_2\rangle.$$

So we find this state by hitting with $\hat{J}_-$, and the state with $j = j_1 + j_2 - 1$ is orthogonal to this state.

Similarly, if we look at $m_j = j_1 + j_2 - 2$, there are three states:

$$\begin{cases} m_1 = j_1,\ m_2 = j_2 - 2 \\ m_1 = j_1 - 1,\ m_2 = j_2 - 1 \\ m_1 = j_1 - 2,\ m_2 = j_2 \end{cases}$$

And we will find the $j = j_1 + j_2 - 2$, $j = j_1 + j_2 - 1$, and $j = j_1 + j_2$ states in this subspace, and so on.

5 Smallest allowed j value

Now suppose $j_1 \geq j_2$ and the biggest m is $j_1 + j_2$. This implies the biggest j is $j = j_1 + j_2$ but the smallest j is not necessarily 0 or $1/2$. It is actually $|j_1 - j_2| = j_{\text{min}}$. We can see this illustrated in the following example.

Example 5.1. Consider the case of when $j_1 = 2$ and $j_2 = 1$. We can draw out all values and scenarios as follows:

m	m_1	m_2	j
3	2	1	$j = 3$
2	2	0	$j = 3, 2$
	1	1	
1	2	-1	$j = 3, 2, 1$
	1	0	
	0	1	
0	1	-1	$j = 3, 2,$ and 1. No 0!
	0	0	
	-1	1	
-1	0	-1	$j = 3, 2, 1$
	-1	0	
	-2	1	
-2	-1	-1	$j = 3, 2$
	-2	0	
-3	-2	-1	$j = 3$

We can also do the following counting check. The total number of states in $|j, m, j_1, j_2\rangle$ representation is given by $\sum_{j=j_{\text{min}}}^{j_{\text{max}}}(2j+1)$. Using that identity $\sum_{j=0}^{m} j = (n+1)n/2$, we get:

$$\sum_{j=j_{\text{min}}}^{j_{\text{max}}} (2j+1) = 2\left(\frac{(j_{\text{max}}+1)j_{\text{max}}}{2} - \frac{j_{\text{min}}(j_{\text{min}}-1)}{2}\right) + j_{\text{max}} - j_{\text{min}} + 1$$
$$= (j_1 + j_2 + 1)(j_1 + j_2) - j_{\text{min}}(j_{\text{min}} - 1) + j_1 + j_2 - j_{\text{min}} + 1$$

However, we also know that in the $|j_1, m_1, j_2, m_2\rangle$ representation $(2j_1 + 1)(2j_2 + 1)$ is the total number of states. Since:

$$(2j_1 + 1)(2j_2 + 1) = 4j_1j_2 + 2j_1 + 2j_2 + 1.$$

Subtracting the two results for the total number of states should give zero:

$$\begin{aligned} 0 &= j_1^2 + 2j_1j_2 + j_2^2 + j_1 + j_2 - j_{\text{min}}(j_{\text{min}} - 1) + j_1 + j_2 - j_{\text{min}} + 1 \\ &- (4j_1j_2 + 2j_1 + 2j_2 + 1) \\ &= j_1^2 - 2j_1j_2 + j_2^2 - j_{\text{min}}(j_{\text{min}}) \end{aligned}$$

As a result, we find $j_{\text{min}}^2 - (j_1 - j_2)^2 = 0$, which implies that:

$$j_{\text{min}} = |j_1 - j_2|.$$

To answer the second question, we can convert between the two representations as follows:

$$|j, m, j_1, j_2\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, m_1, j_2, m_2\rangle\langle j_1, m_1, j_2, m_2|j, m, j_1, j_2\rangle,$$

where $|j_1, m_1, j_2, m_2\rangle\langle j_1, m_1, j_2, m_2|j, m, j_1, j_2\rangle$ are our Clebsch-Gordan coefficients. Similarly:

$$|j_1, m_1, j_2, m_2\rangle = \sum_{j=j_1+j_2}^{j=j_1-j_2} \sum_{m=-j}^{j} |j, m, j_1, j_2\rangle\langle j, m, j_1, j_2|j_1, m_1, j_2, m_2\rangle,$$

where $\langle j, m, j_1, j_2|j_1, m_1, j_2, m_2\rangle$ are the complex conjugates of the Clebsch-Gordan coefficients above, which are also called Clebsch-Gordan coefficients. Note that the phases are chosen so that coefficients are real, so the only ambiguity is in the $\pm$ signs, which are fixed by a convention.

6 Summary

Once we have identified the allowed j values, we can compute every state in the coupled basis in terms of the uncoupled basis by using the lowering operators and orthogonality. This procedure becomes very tedious to work with and we might want to seek a simpler way to do it. There is no free lunch, but there are alternatives one can use to do this. Often, if one needs to combine many angular momenta together, a computer can be employed to do the algebra for you and save a lot of effort.