Phys 506 lecture 13: Addition of Angular Momentum II

1 Two Spin-$\frac{1}{2}$ Particles

Let's examine the simplest case concretely: $j_1 = \frac{1}{2}$ and $j_2 = \frac{1}{2}$.

- We can form a $J = 1$ state (called a *triplet*) and
- We can form a $J = 0$ state (called a *singlet*)

Obviously $j_1 = j_2 = \frac{1}{2}$ always. Then $|j, m_j, j_1, j_2\rangle = |j, m_j\rangle$ can be written as

$$|j = 1, m_j = +1\rangle = \left|m_1 = +\frac{1}{2}, m_2 = +\frac{1}{2}\right\rangle = |\uparrow\uparrow\rangle\,,$$

since this is the only way to get $+1$ for m_j and

$$|j = 1, m_j = -1\rangle = \left|m_1 = -\frac{1}{2}, m_2 = -\frac{1}{2}\right\rangle = |\downarrow\downarrow\rangle\,,$$

but for $m_j = 0$ we have two possibilities

$$\begin{aligned}|j = 1, m_j = 0\rangle &= \alpha\left|m_1 = +\frac{1}{2}, m_2 = -\frac{1}{2}\right\rangle + \beta\left|m_1 = -\frac{1}{2}, m_2 = +\frac{1}{2}\right\rangle \\ &= \alpha\,|\uparrow\downarrow\rangle + \beta\,|\downarrow\uparrow\rangle\,.\end{aligned}$$

How do we find α and β?

Answer: Use the total spin lowering operator. We know

$$\begin{aligned}J^-\,|j = 1, m_j = 1\rangle &= \hbar\sqrt{(j + m_j)(j - m_j + 1)}\,|j = 1, m_j = 0\rangle \\ &= \hbar\sqrt{2\cdot 1}\,|j = 1, m_j = 0\rangle \\ &= \sqrt{2}\hbar\,|j = 1, m_j = 0\rangle\end{aligned}$$

But $J^- = J_1^- + J_2^- = S_1^- + S_2^-$ where $S^-\,|\uparrow\rangle = \hbar\,|\downarrow\rangle$. So,

$$\begin{aligned}(S_1^- + S_2^-)\,|\uparrow\uparrow\rangle &= \hbar\,|\downarrow\uparrow\rangle + \hbar\,|\uparrow\downarrow\rangle \\ &= \sqrt{2}\hbar\,|j = 1, m_j = 0\rangle\end{aligned}$$

which tells us that $\alpha = \beta = \frac{1}{\sqrt{2}}$. Hence,

$$\boxed{|j = 1, m_j = 0\rangle = \frac{1}{\sqrt{2}}\,|\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}\,|\downarrow\uparrow\rangle}$$

How to find $|j=0,m_j=0\rangle$? Well, it must be orthogonal to $|j=1,m_j=0\rangle$, since

$$\langle j=1,m_j=0|j=0,m_j=0\rangle = 0,$$

so it must be

$$\boxed{|j=0,m_j=0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle}.$$

Note that we have an ambiguity of a ± 1 for the state.

2 Spin-$\frac{1}{2}$ and arbitrary j

Here, we consider a general case where j_1 is arbitrary and $j_2 = \frac{1}{2}$. Then,

$$j = j_1 + \frac{1}{2} \text{ or } j_1 - \frac{1}{2}$$

Obviously,

$$\left|j = j_1 + \frac{1}{2}, m_j = j_1 + \frac{1}{2}\right\rangle = \left|m_1 = j_1, m_2 = \frac{1}{2}\right\rangle.$$

To find $\left|j=j_1+\frac{1}{2}, m_j = j_1 - \frac{1}{2}\right\rangle$, use the lowering operator, recalling that

$$\begin{aligned}
J^- \left|j = j_1 + \frac{1}{2}, m_j = j_1 + \frac{1}{2}\right\rangle &= \hbar\sqrt{\left(j_1 + \frac{1}{2} + j_1 + \frac{1}{2}\right)\left(j_1 + \frac{1}{2} - j_1 - \frac{1}{2} + 1\right)}\left|j = j_1 + \frac{1}{2}, m_j = j_1 - \frac{1}{2}\right\rangle \\
&= \hbar\sqrt{2j_1+1}\left|j = j_1 + \frac{1}{2}, m_j = j_1 - \frac{1}{2}\right\rangle.
\end{aligned}$$

But $J^- = J_1^- + S_2^-$ so

$$J^- \left|j_1, m_2 = \frac{1}{2}\right\rangle = \hbar\sqrt{2j_1}\left|m_1 = j_1 - 1, m_2 = \frac{1}{2}\right\rangle + \hbar\left|m_1 = j_1, m_2 = \frac{1}{2}\right\rangle.$$

So

$$\boxed{\left|j = j_1 + \frac{1}{2}, m_j = j_1 - \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2j_1+1}}\left|m_1 = j_1, m_2 = -\frac{1}{2}\right\rangle + \sqrt{\frac{2j_1}{2j_1+1}}\left|m_1 = j_1 - 1, m_2 = \frac{1}{2}\right\rangle}.$$

The state $\left|j_1 = j_1 - \frac{1}{2}, m_j = j_1 - \frac{1}{2}\right\rangle$ is orthogonal to this so

$$\boxed{\left|j_1 = j_1 - \frac{1}{2}, m_j = j_1 - \frac{1}{2}\right\rangle = \sqrt{\frac{2j_1}{2j_1+1}}\left|m_1 = j_1, m_2 = -\frac{1}{2}\right\rangle - \frac{1}{\sqrt{2j_1+1}}\left|m_1 = j_1 - 1, m_2 = \frac{1}{2}\right\rangle.}$$

How do we find lower m_j values? Use the lowering operator again.

$$\begin{aligned}
\left|j = j_1 + \frac{1}{2}, m_j = j_1 - \frac{3}{2}\right\rangle &= \frac{1}{\hbar\sqrt{2j_1 \cdot 2}} J^- \left|j = j_1 + \frac{1}{2}, m_j = j_1 - \frac{1}{2}\right\rangle \\
&= \frac{1}{\sqrt{2j_1 \cdot 2(2j_1+1)}} \cdot \sqrt{2j_1 \cdot 1}\left|m_1 = j_1 - 1, m_2 = -\frac{1}{2}\right\rangle \\
&+ \frac{1}{\sqrt{2j_1 \cdot 2(2j_1+1)}}\sqrt{2j_1}\sqrt{(2j_1)2}\left|m_1 = j_1 - 2, m_2 = \frac{1}{2}\right\rangle \\
&+ \frac{1}{\sqrt{2j_1 \cdot 2(2j_1+1)}}\sqrt{2j_1}\left|m_1 = j_1 - 1, m_2 = -\frac{1}{2}\right\rangle.
\end{aligned}$$

Therefore,

$$\boxed{\left|j = j_1 + \frac{1}{2}, m_j = j_1 - \frac{3}{2}\right\rangle = \sqrt{\frac{2}{2j_1+1}}\left|m_1 = j_1 - 1, m_2 = -\frac{1}{2}\right\rangle + \sqrt{\frac{2j_1-1}{2j_1+1}}\left|m_1 = j_1 - 2, m_2 = \frac{1}{2}\right\rangle.}$$

In general, one finds

$$\left|j = j_1 + \frac{1}{2}, m_j\right\rangle = \sqrt{\frac{j_1 - m_j + \frac{1}{2}}{2j_1+1}}\left|m_1 = m_j + \frac{1}{2}, m_2 = -\frac{1}{2}\right\rangle + \sqrt{\frac{j_1 + m_j + \frac{1}{2}}{2j_1+1}}\left|m_1 = m_j - \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle.$$

Similarly,

$$\left|j = j_1 - \frac{1}{2}, m_j\right\rangle = \sqrt{\frac{j_1 + m_j + \frac{1}{2}}{2j_1+1}}\left|m_1 = m_j + \frac{1}{2}, m_2 = -\frac{1}{2}\right\rangle - \sqrt{\frac{j_1 - m_j + \frac{1}{2}}{2j_1+1}}\left|m_1 = m_j - \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle.$$

3 Alternative Derivation

Another way to derive this is as follows. Start with a general expression for the state:

$$|j, m_j\rangle = \alpha\left|m_1 = m_j - \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle + \beta\left|m_1 = m_j + \frac{1}{2}, m_2 = -\frac{1}{2}\right\rangle.$$

Using the spherical basis for the dot product, write the total squared angular momentum as

$$\begin{aligned}\hat{J}^2 = (\hat{J}_1 + \hat{J}_2)^2 &= \hat{J}_1^2 + \hat{J}_2^2 + 2\hat{J}_1 \cdot \hat{J}_2 \\ &= \hat{J}_1^2 + \hat{J}_2^2 + 2\left(\frac{J_1^+ J_2^- + J_1^- J_2^+}{2}\right) + 2J_1^z J_2^z.\end{aligned}$$

Then, act $\hat{J}^2$ on the state and use the known results for the different operators in the uncoupled basis:

$$\begin{aligned}\frac{\hat{J}^2}{\hbar^2}|j, m_j\rangle = \alpha&\left|m_1 = m_j - \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle\left(j_1(j_1+1) + \frac{3}{4} + 2(m_j - \frac{1}{2})\cdot\frac{1}{2}\right) \\ &+ \alpha\left|m_1 = m_j + \frac{1}{2}, m_2 = -\frac{1}{2}\right\rangle\left(\sqrt{(j_1 - m_j + \frac{1}{2})(j_1 + m_j - \frac{1}{2} + 1)}\right) \\ &+ \beta\left|m_1 = m_j + \frac{1}{2}, m_2 = -\frac{1}{2}\right\rangle\left(j_1(j_1+1) + \frac{3}{4} + 2(m_j + \frac{1}{2})(-\frac{1}{2})\right) \\ &+ \beta\left|m_1 = m_j - \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle\left(\sqrt{(j_1 + m_j + \frac{1}{2})(j_1 - m_j - \frac{1}{2} + 1)}\right) \\ &= \begin{pmatrix} j_1(j_1+1) + m_j + \frac{1}{4} & \sqrt{(j_1 - m_j + \frac{1}{2})(j_1 + m_j + \frac{1}{2})} \\ \sqrt{(j_1 + m_j + \frac{1}{2})(j_- + m_j + \frac{1}{2})} & j_1(j_1+1) - m_j + \frac{1}{4}\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}\end{aligned}$$

We want this to be proportional to $\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ to be an eigenstate of $\hat{J}^2$.

So,

$$\det\begin{pmatrix} j_1(j_1+1)+m_j+\frac{1}{4}-\lambda & \sqrt{(j_1-m_j+\frac{1}{2})(j_1+m_j+\frac{1}{2})} \\ \sqrt{(j_1+m_j+\frac{1}{2})(j_-+m_j+\frac{1}{2})} & j_1(j_1+1)-m_j+\frac{1}{4}-\lambda \end{pmatrix}=0,$$

$$\implies \lambda^2+\lambda\left(-2(j_1+\frac{1}{2})^2\right)+\left(j_1+\frac{1}{2}\right)^4-m_j^2-\left(j_1+\frac{1}{2}\right)^2+m_j^2=0.$$

Simplifying, we get

$$\lambda^2-2\left(j_1+\frac{1}{2}\right)^2\lambda+\left(j_1+\frac{1}{2}\right)^2\left(1+\left(j_1+\frac{1}{2}\right)^2\right)=0.$$

Solving for λ, we find that

$$\lambda=\left(j_1+\frac{1}{2}\right)^2\pm\frac{1}{2}\sqrt{4\left(j_1+\frac{1}{2}\right)^4-4\left(j_1+\frac{1}{2}\right)^4+4\left(j_1+\frac{1}{2}\right)^2}$$

$$\implies \left(j_1+\frac{1}{2}\right)\left(j_1+\frac{1}{2}\pm 1\right).$$

One root is $j=j_1+\frac{1}{2}$ and the other is $j_1-\frac{1}{2}$. Note that we knew this ahead of time, so we did not really need to solve the equation. We now need to find α and β for each. $\lambda=(j+1/2)(j_1+1/2\pm 1)$ so,

$$\left(\left(j+\frac{1}{2}\right)^2-\left(j+\frac{1}{2}\right)\left(j+\frac{1}{2}\pm 1\right)+m_j\right)\alpha+\sqrt{\left(j+\frac{1}{2}\right)^2-m^2}\beta=0$$

$$\left(\mp\left(j+\frac{1}{2}\right)+m_j\right)\alpha+\sqrt{\left(j+\frac{1}{2}\right)^2-m^2}\beta=0.$$

Therefore,

$$\beta=\frac{\pm\left(j+\frac{1}{2}\right)-m_j}{\sqrt{\left(j+\frac{1}{2}\right)^2-m_j^2}}\alpha=\pm\sqrt{\frac{j_1\mp m_j+\frac{1}{2}}{j_1\pm m_j+\frac{1}{2}}}\alpha.$$

Now, let

$$\alpha=C\sqrt{j_1\pm m_j+\frac{1}{2}}$$

$$\beta=C\sqrt{j_1\mp m_j+\frac{1}{2}}$$

Then, we must demand that $\alpha^2+\beta^2=1$

$$C^2\left(j_1\pm m_j+\frac{1}{2}+j_1\mp m_j+\frac{1}{2}\right)=1\implies C=\frac{1}{\sqrt{2j_1+1}}.$$

Hence,

$$\left|j_1+\frac{1}{2},m_j\right\rangle=\sqrt{\frac{j_1+m_j+\frac{1}{2}}{2j_1+1}}\left|m_1=m_j-\frac{1}{2},m_2=\frac{1}{2}\right\rangle+\sqrt{\frac{j_1-m_j+\frac{1}{2}}{2j_1+1}}\left|m_1=m_j+\frac{1}{2},m_2=-\frac{1}{2}\right\rangle$$

$$\left|j_1-\frac{1}{2},m_j\right\rangle=-\sqrt{\frac{j_1-m_j+\frac{1}{2}}{2j_1+1}}\left|m_1=m_j-\frac{1}{2},m_2=\frac{1}{2}\right\rangle+\sqrt{\frac{j_1+m_j+\frac{1}{2}}{2j_1+1}}\left|m_1=m_j+\frac{1}{2},m_2=-\frac{1}{2}\right\rangle.$$