# Phys 506 lecture 2: The Operator Identities

It turns out that there are five critically important operates identities that we need to know to do just about anything in quantum mechanics.

They are the Leibniz (or product rule) identity, the Hadamard lemma (and braiding identity), the exponential reordering identity, the Baker-Campbell-Hausdorff identity and the exponential disentangling identity.

All but one of them are quite elementary to prove. One is hard. Really hard. So we only do it for special cases.

## 1 Leibniz product rule

1.) The Leibniz or product rule

$$[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$$

Proof:

$$\begin{aligned}
[\hat{A}, \hat{BC}] &= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} \\
\text{add } 0 &= \hat{A}\hat{B}\hat{C} \underbrace{- \hat{B}\hat{A}\hat{C} + \hat{B}\hat{A}\hat{C}}_{=0} - \hat{B}\hat{C}\hat{A} \\
\text{regroup} &= (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A}) \\
&= [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]
\end{aligned}$$

It acts just like a differentiation product rule, where the single operator in the commutator is like a derivative and the product of operators is like a product of functions. This comparison is actually very deep, as one can define derivatives via the Leibniz rule and avoid having to use limits. I won't go into the details of that here though.

It has other forms too, like

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{\vec{B}}, \hat{C}]\hat{B}$$

.

Related to this is the Jacobi identity

$$[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0$$

which I don't find I use very much. This can be proven just by writing it out

$$
\begin{aligned}
&\hat{A}[\hat{B},\hat{C}]-[\hat{B},\hat{C}]\hat{A}+\hat{B}[\hat{C},\hat{A}]-[\hat{C},\hat{A}]\hat{B}+\hat{C}[\hat{A},\hat{B}]-[\hat{A},\hat{B}]\hat{C}\\
=&\hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}-\hat{B}\hat{C}\hat{A}+\hat{C}\hat{B}\hat{A}+\hat{B}\hat{C}\hat{A}-\hat{B}\hat{A}\hat{C}\\
&-\hat{C}\hat{A}\hat{B}+\hat{A}\hat{C}\hat{B}+\hat{C}\hat{A}\hat{B}-\hat{C}\hat{B}\hat{A}-\hat{A}\hat{B}\hat{C}+\hat{B}\hat{A}\hat{C}\\
=&0
\end{aligned}
$$

## 2 Hadamard lemma

2.) Next up is the Hadamard lemma:

$$
\begin{aligned}
e^{\hat{A}}\hat{B}e^{-\hat{A}}&=\hat{B}+[\hat{A},\hat{B}]+\frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]]+\cdots\\
&=\sum_{n=0}^{\infty}\underbrace{[\cdots(\hat{A},\hat{B}]\cdots]_n}_{\text{n-fold nested commutator}}\ \frac{1}{n!}
\end{aligned}
$$

Proof: Consider

$$
\begin{aligned}
f(x)&=e^{x\hat{A}}\hat{B}e^{-x\hat{A}}\\
f(0)&=\hat{B}\\
f'(0)&=\hat{A}e^{x\hat{A}}\hat{B}e^{-x\hat{A}}-e^{x\hat{A}}\hat{B}e^{-x\hat{A}}\hat{A}\Big|_{x=0}\\
&=[\hat{A},\hat{B}]\\
f''(0)&=\hat{A}f'(0)-f'(0)\hat{A}\\
&=[\hat{A},[\hat{A}\mid\hat{B}]].
\end{aligned}
$$

Now, in general, if we assume

$$
\begin{aligned}
f^{(n)}(0)&=[\hat{A},\cdots[\hat{A},\hat{B}]\cdots]_n\\
\text{then } f^{(n+1)}(0)&=\hat{A}f^{(n)}(0)-f^{(n)}(0)\hat{A}\\
&=[\hat{A},\cdots[\hat{A},\hat{B}]\cdots]_{n+1}.
\end{aligned}
$$

Then using Taylor's theorem with $x=1$ gives us the proof for the Hadamard lemma:

$$
\boxed{e^{\hat{A}}\hat{B}e^{-\hat{A}}=\sum_{n=0}^{\infty}\frac{1}{n!}[\hat{A},[\cdots[\hat{A},\hat{B}]\cdots]_n.}
$$

The braiding identity is even more powerful. It comes from the fact that

$$
\begin{aligned}
e^{\hat{A}}\hat{B}^ne^{-\hat{A}}&=\underbrace{e^{\hat{A}}\hat{B}e^{-\hat{A}}e^{\hat{A}}\hat{B}e^{-\hat{A}}\cdots e^{\hat{A}}\hat{B}e^{-\hat{A}}}_{n\text{ triples}}\\
&=(e^{\hat{A}}\hat{B}e^{-\hat{A}})^n.
\end{aligned}
$$

Hence, any operator series (or any function that an be expressed as a series) satisfies

$$f(\hat{B}) = \sum_n c_n \hat{B}^n.$$

then

$$\begin{aligned} e^{\hat{A}} f(\hat{B}) e^{-\hat{A}} &= \sum_n c_n e^{\hat{A}} \hat{B}^n e^{-\hat{A}} = \sum_n c_n (e^{\hat{A}} \hat{B} e^{-\hat{A}})^n \\ &= f\left(e^{\hat{A}} \hat{B} e^{-\hat{A}}\right) \end{aligned}$$

so

$$\boxed{\begin{aligned} e^{\hat{A}} f(\hat{B}) e^{-\hat{A}} &= f\left(e^{\hat{A}} \hat{B} e^{-\hat{A}}\right) \\ &= f\left(\sum_{n=0}^{\infty} \frac{1}{n!} [\hat{A}, \cdots [\hat{A}, \hat{B}] \cdots ]_n \right) \end{aligned}}$$

This is the braiding relation.

# 3 Exponential re-ordering identity

3.) The exponential reordering identity is most powerful in simple cases as we will see.

If I have $e^{\hat{A}} e^{\tilde{B}}$ and I want to interchange the order, what do I get?
In other words, $e^{\hat{A}} e^{\hat{B}} =$ what $e^{\hat{A}}$ ? The way to solve that is with braiding:

$$\begin{aligned} e^{\hat{A}} e^{\hat{B}} e^{-\hat{A}} &= e^{\left(e^{\hat{A}} \hat{B} e^{-\hat{A}}\right)} \\ &= e^{\hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2}[\hat{A}, [\hat{A}, \hat{B}]] + \cdots}. \end{aligned}$$

Multiply both sides by $e^{\hat{A}}$ on the right

$$\boxed{e^{\hat{A}} e^{\hat{B}} = e^{\hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2}[\hat{A}, [\hat{A}, \hat{B}]] + \cdots} e^{\hat{A}}}$$

In general, this is a complicated mess, but what if $[\hat{A}, [\hat{A}, \hat{B}]] = 0$, like what happens if $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$ ?
Then, we get

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{B} + [\hat{A}, \hat{B}]} e^{\hat{A}}$$

But if $[\hat{B}, [\hat{A}, \hat{B}]] = 0$ too (as it would in our example) then we have

$$\boxed{\begin{gathered} e^{\hat{A}} e^{\hat{B}} = e^{[\hat{A}, \hat{B}]} e^{\hat{B}} e^{\hat{A}} \\ \text{when } [\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0 \end{gathered}}$$

This is a very powerful identity that comes up often.

# 4 Baker-Campbell-Hausdorff formula

4.) We are now ready for the hard identity, Baker-Campbell- Hausdorff. It asks the question

$$e^{\hat{A}}e^{\hat{B}} = e^{\text{what ?}} \ ,$$

which is kind of half way to the exponential re-ordering identity. But it is much harder to work out. In fact, it leads to an infinite series of commutators, just like the Hadamard lemma, but the coefficients have no regular structure, even though one can find equations that determine them.

As a warmup, let's work of the case when $[\hat{A}, \hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$. This is called the Weyl form of the identity.

First define

$$\begin{aligned}
&f(\lambda) = e^{-\lambda\hat{A}}e^{\lambda(\hat{A}+\hat{B})}e^{-\lambda\hat{B}} \\
&\frac{df(\lambda)}{d\lambda} = -\hat{A}e^{-\lambda\hat{A}}e^{\lambda(\hat{A}+\hat{B})}e^{-\lambda\hat{B}} + e^{-\lambda\hat{A}}(\hat{A}+\hat{B})e^{\lambda(\hat{A}+\hat{B})}e^{-\lambda\hat{B}} \\
&- e^{-\lambda\hat{A}}e^{\lambda(\hat{A}+\hat{B})}e^{-\lambda\hat{B}}\hat{B}
\end{aligned}$$

But $\hat{A}e^{-\lambda\hat{A}} = e^{-\lambda\hat{A}}\hat{A}$ and $\hat{B}e^{\lambda(\hat{A}+\hat{B})}$ can be rewritten with Hadamard as follows:

$$\begin{aligned}
&e^{\lambda(\hat{A}+\hat{B})}e^{-\lambda(\hat{A}+\hat{B})}\hat{B}e^{\lambda(\hat{A}+\hat{B})} \\
&= e^{\lambda(\hat{A}+\hat{B})}(\hat{B} - \lambda[\hat{A}+\hat{B}, \hat{B}] + \frac{\lambda^2}{2}[\hat{A}+\hat{B}, [\hat{A}+\hat{B}, \hat{B}]]. \\
&= e^{\lambda(\hat{A}+\hat{B})}(\hat{B} - \lambda[\hat{A}, \hat{B}] + 0)
\end{aligned}$$

$$\text{So } \frac{df(\lambda)}{d\lambda} = -\lambda[\hat{A}, \hat{B}]f(\lambda) \Rightarrow f(\lambda) = e^{-\frac{\lambda^2}{2}[\hat{A},\hat{B}]}f(0)$$

or

$$e^{-\frac{\lambda^2}{2}(\hat{A},\hat{B}]} = e^{-\lambda\hat{A}}e^{\lambda(\hat{A}+\hat{B})}e^{-\lambda\hat{B}}.$$

Multiply on left with $e^{\lambda\hat{A}}$ and on right with $e^{\lambda\hat{B}}$ to get

$$e^{\lambda\hat{A}} \qquad \underbrace{e^{-\frac{\lambda^2}{2}[\hat{A},\hat{B}]}}_{\text{commutes with everything}} \qquad e^{\lambda\hat{B}} = e^{\lambda(\hat{A}+\hat{B})}$$

So $e^{\lambda\hat{A}}e^{\lambda\hat{B}} = e^{\lambda(\hat{A}+\hat{B})+\frac{\lambda^2}{2}[\hat{A},\hat{B}]}$. Now set $\lambda = 1$ to obtain the final result

$$\boxed{\begin{gathered} e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]} \\ \text{when } [\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}[\hat{A}, \hat{B}]] = 0 \end{gathered}}$$

This is the easy case of BCH and the version most people know.

Now we work out the general case! We need another identity (called Sneddon's formula)

$$\frac{d}{dx}e^{\hat{A}(x)} = \int_0^1 dye^{(1-y)\hat{A}(x)}\frac{d\hat{A}(x)}{dx}e^{y\hat{A}(x)}.$$

This looks weird, but it takes this form because

$$\left[\hat{A}(x), \frac{d\hat{A}}{dx}\right] \neq 0 \text{ in general.}$$

Recall that with operators, we have $\frac{d}{dx}e^{\hat{A}(x)} = \frac{d}{dx}\left[1 + \hat{A} + \frac{1}{2}\hat{A}^2 + \frac{1}{6}\hat{A}^3 + \cdots\right]$

$$= \hat{A}' + \frac{1}{2}(\hat{A}'\hat{A} + \hat{A}\hat{A}') + \frac{1}{6}(\hat{A}'\hat{A}^2 + \hat{A}\hat{A}'\hat{A} + \hat{A}^2\hat{A}')\cdots$$

because we cannot switch orders of the derivative and the operator. The $n$th order terms in the series will have $(m)$ powers of $\hat{A}\hat{A}'$ and $(n-m-1)$ powers of $\hat{A}$; they are divided by $n!$. We can rewrite this as

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n+m+1)!}\hat{A}^n\hat{A}'\hat{A}^m$$

Now expand each term in the integrand

$$\int_0^1 dy \sum_{n=0}^{\infty}\frac{(1-y)^n\hat{A}^n}{n!}\hat{A}'\sum_{m=0}^{\infty}\frac{y^m}{m!}\hat{A}^m$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\hat{A}^n\hat{A}'\hat{A}^m}{n!m!}\underbrace{\int_0^1 dy(1-y)^n y^m}_{\text{Beta function } \beta(m,n)}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\hat{A}^n\hat{A}'\hat{A}^m}{n!m!}\cdot\frac{n!m!}{(n+m+1)!}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\hat{A}^n\hat{A}'\hat{A}^m}{(n+m+1)!}.$$

So

$$\frac{d}{dx}e^{\hat{A}} = \int_0^1 dy e^{(1-y)\hat{A}}\hat{A}'e^{y\hat{A}}.$$

Now, multiply on the left by $e^{-\hat{A}}$ to get

$$e^{-\hat{A}}\frac{d}{dx}e^{\hat{A}} = \int_0^1 dy\,\underbrace{e^{-y\hat{A}}\hat{A}'e^{y\hat{A}}}_{\text{Hadamard}}$$
$$= \int_0^1 dy(\hat{A}' - y[\hat{A},\hat{A}'] + \frac{y^2}{2}[\hat{A},[\hat{A},\hat{A}']] + \cdots)$$
$$= \hat{A}' + \frac{1}{2}\left[\hat{A}',\hat{A}\right] + \frac{1}{6}\left[\left(\hat{A}',\hat{A}\right],\hat{A}\right] + \cdots.$$

To obtain the general form of BCH, we define

$$e^{x\hat{A}}e^{x\hat{B}} = e^{\hat{G}(x)} = e^{x\hat{G}_1 + x^2\hat{G}_2 + x^3\hat{G}_3 + \cdots}$$
$$e^{-x\hat{B}}e^{-x\hat{A}}\frac{d}{dx}\left(e^{x\hat{A}}e^{x\hat{B}}\right) = e^{-\hat{G}(x)}\frac{d}{dx}e^{\hat{G}(x)}$$

The LHS can be evaluated directly [ Note $e^{-x\hat{B}}e^{-x\hat{A}} = e^{-\hat{G}[x)}$ ]

$$\begin{aligned}
&e^{-x\hat{B}}e^{-x\hat{A}}\frac{d}{dx}\left(e^{x\hat{A}}e^{x\hat{B}}\right)\\
&= e^{-x\hat{B}}e^{-x\hat{A}}\left(\hat{A}e^{x\hat{A}}e^{x\hat{B}} + e^{x\hat{A}}e^{x\hat{B}}\hat{B}\right)\\
&= \hat{B} + e^{-x\hat{B}}\hat{A}e^{x\hat{B}}\\
&= \hat{B} + \hat{A} - x[\hat{B},\hat{A}] + \frac{x^2}{2}[\hat{B},[\hat{B},\hat{A}]] + \cdots
\end{aligned}$$

But we also have $e^{-\hat{G}(x)}\frac{d}{dx}e^{\hat{G}(x)} = G^1 + \frac{1}{2}[\hat{G}', G] + \frac{1}{6}[[G, G'], G] + \cdots$
Use a power series for $\hat{G}(x) = x\hat{G}_1 + x^2\hat{G}_2 + x^3\hat{G}_3 + \cdots$

$$\begin{aligned}
G' &= \hat{G}_1 + 2x\hat{G}_2 + 3x^2\hat{G}_3 + \cdots\\
\frac{1}{2}[G', G] &= \frac{1}{2}[\hat{G}_1 + 2x\hat{G}_2 + 3x^2\hat{G}_3 + \cdots, x\hat{G}_1 + x^2\hat{G}_2 + x^3\hat{G}_3\cdots]\\
&= \frac{1}{2}(x^2[\hat{G}_1,\hat{G}_2] + 2x^2[\hat{G}_2,\hat{G}_1]\\
&+ x^3[\hat{G}_1,\hat{G}_3] + 3x^3[\hat{G}_3,\hat{G}_1]) + \cdots\\
&= \frac{1}{2}x^2[\hat{G}_2,\hat{G}_1] + x^3[\hat{G}_3,\hat{G}_1] + \cdots\\
\frac{1}{6}[[G', G], G] &= \frac{1}{6}[[\hat{G}_1, x^2\hat{G}_2], x\hat{G}_1] = \frac{x^3}{6}[[\hat{G}_1,\hat{G}_2],\hat{G}_1]
\end{aligned}$$

So we have

$$e^{-\hat{G}(x)}\frac{d}{dx}e^{\hat{G}(x)} = \hat{G}_1 + 2x\hat{G}_2 + 3x^2\hat{G}_3 + \frac{x^2}{2}[\hat{G}_2\hat{G}_1] + O(x^3),$$

but this equals

$$\hat{A} + \hat{B} + x[\hat{A},\hat{B}] + \frac{x^2}{2}[\hat{B},[\hat{B},\hat{A}]] + \cdots.$$

So

$$\begin{aligned}
G_1 &= \hat{A} + \hat{B}\\
G_2 &= \frac{1}{2}[\hat{A},\hat{B}]\\
3G_3 + \frac{1}{2}[G_2, G_1] &= \frac{1}{2}[\hat{B},[\hat{B},\hat{A}]]\\
G_3 &= \frac{1}{6}[\hat{B},[\hat{B},\hat{A}]] - \frac{1}{6}\left[\frac{1}{2}[\hat{A},\hat{B}], \hat{A} + \hat{B}\right]\\
&= \frac{1}{6}[\hat{B},[\hat{B},\hat{A}]] + \frac{1}{12}[\hat{A},[\hat{A},\hat{B}]] - \frac{1}{12}[\hat{B},[\hat{B},\hat{A}]]\\
G_3 &= \frac{1}{12}[\hat{B},[\hat{B},\hat{A}]] + \frac{1}{12}[\hat{A},[\hat{A},\hat{B}]].
\end{aligned}$$

But if we set $(x = 1)$, then

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{G}(x=1)} = e^{\hat{G}_1 + \hat{G}_2 + \hat{G}_3 + \cdots}$$

$$\text{So } \boxed{e^{\hat{A}}e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2}[\hat{A},\hat{B}] + \frac{1}{12}[\hat{A},[\hat{A},\hat{B}]) + \frac{1}{12}[\hat{B},(\hat{B},\hat{A}]) + \ldots}}$$

which is the BCH. We can work out higher order terms one by one, but it is painful.

# 5 Exponential disentangling identity

5.) The fifth identity is the exponential disentangling and it uses a theorem from Lie algebras:
Any identity expressed in terms of exponentials of the generators of a Lie algebra that is proved for one faithful representation holds in all representations.

So, the identity we worked at last time

$$\exp\left[i\frac{2\vec{v}\cdot\vec{s}}{\hbar}\right] = \exp\left[\alpha\frac{2\hat{S}_+}{\hbar}\right]\exp\left[\beta\frac{2\hat{S}_z}{\hbar}\right]\exp\left[\gamma\frac{2\hat{S}_-}{\hbar}\right]$$

with $\alpha$, $\beta$, and $\gamma$ determined as before, holds for any angular momentum representation!

This is amazingly powerful. We proved something about Pauli spin matrices and it holds or all $j$!! We will use this again later.