Phys 506 lecture 23: 3D scattering continued

1 Wigner threshold law

Last time, we showed from the delta shell solution that at low energy $\tan\delta_\ell(k) \sim (ka)^{2\ell+1}$. This is called the **Wigner Threshold Law**, and it shows that s-wave scattering dominates at low energy. We parametrize low-energy scattering with a **scattering length** a_0 and an **effective range** r_0:

$$k^{2\ell+1}\cot\delta_\ell(k) = -\frac{1}{a_\ell} + \frac{1}{2}k^2 r_\ell^2 + \mathcal{O}(k^4)$$

where a_ℓ and r_ℓ are constants.

Now recall, that we found for the delta shell potential:

$$\tan\delta_\ell(k) = \frac{\lambda a (ka)^{2\ell+1}}{[(2\ell+1)!!]^2\left(1-\frac{\lambda a}{2\ell+1}\right)}.$$

Hence:

$$k^{2\ell+1}\cot\delta_\ell(k) = \frac{[(2\ell+1)!!]^2\left(1-\frac{\lambda a}{2\ell+1}\right)}{\lambda a}$$

which implies:

$$a_\ell = -\frac{\lambda a}{1-\frac{\lambda a}{2\ell+1}}$$

and so the scattering length is:

$$a_0 = -\frac{\lambda a}{1-\lambda a}$$

Note that the scattering length goes through a divergence when $\lambda a = 1$. For higher partial waves ($\ell > 0$), divergences occur when $\lambda a = 2\ell+1$. Hence, the scattering length can become much larger than the range of the potential. This phenomenon leads to resonances, where the system becomes highly sensitive to the interaction strength. These resonances are known as **Feshbach resonances** and occur when tuned by an external magnetic field and are very important in ultra-cold atomic physics as they can govern the interactions between atoms and allow one to tune different interactions.

2 Low-Energy Scattering Cross Section

As $k \to 0$, the scattering cross section $\sigma(k)$ is given by:

$$\sigma(k) \to 4\pi \frac{\sin^2 \delta_0(k)}{k^2}$$

But for small k, $\delta_0(k) \sim \frac{\lambda a\, ka}{1-\lambda a} \sim -a_0\, k$. Thus, the low-energy cross section becomes:

$$\sigma(k \to 0) = 4\pi a_0^2$$

This demonstrates that the scattering length a_0 determines the total cross section for low-energy scattering.

3 Origin of the scattering resonance

But what is the origin of the scattering resonance? The scattering resonance is related to the bound states of the potential. Therefore, let's examine the attractive delta shell potential, which has bound states:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} f_{k\ell}(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} f_{k\ell}(r) - \frac{\hbar^2 \lambda}{2m}\delta(r-a) f_{k\ell}(r) = E_{n\ell} f_{k\ell}(r)$$

We already observed in the plane-wave case that the solutions for $r \neq a$ are $j_\ell(ikr)$, $\eta_\ell(ikr)$ where $k^2 = \frac{2mE}{\hbar^2}$ and these are analytically continued to imaginary argument. We also matched the wave functions at $r = a$ and took into account the discontinuity at $r = a$

$$f_\ell(r) = \begin{cases} A j_\ell(ik_1 r) & r < a \\ B j_\ell(ik_2 r) + C\eta_\ell(ik_2 r) & r > a \end{cases}$$

As $r \to 0$, the behavior of $j_\ell(kr)$ is fine since $j_\ell(kr) \sim (kr)^\ell$. For large r:

$$\begin{cases} j_\ell(kr) \sim \frac{1}{ikr} \sin\left(ikr - \ell\frac{\pi}{2}\right) \\ n_\ell(kr) \sim -\frac{1}{kr} \cos\left(ikr - \ell\frac{\pi}{2}\right) \end{cases}$$

To ensure exponential decay, we need $C = iB$. Using the complex identities for sine and cosine:

$$\begin{cases} \cos x = \frac{e^{ix}+e^{-ix}}{2} \\ \sin x = \frac{e^{ix}-e^{-ix}}{2i} \end{cases}$$

we get:

$$\frac{i}{2}(\cos x + \sin x) \sim e^{ix}$$

Setting $x = ikr$ gives us the form of the exponential decay we want. Thus:

$$f_\ell(r) = \begin{cases} A j_\ell(ikr) & r < a, \\ B\left[j_\ell(ikr) + i\eta_\ell(ikr)\right] & r > a. \end{cases}$$

At $r = a$, we impose continuity of $f_\ell(r)$, giving us:

$$Aj_\ell(ika) = B\left[j_\ell(ika) + i\eta_\ell(ika)\right].$$

Rearranging this, we get:

$$A = B\left(1 + i\frac{\eta_\ell(ika)}{j_\ell(ika)}\right).$$

For the derivative discontinuity at $r = a$:

$$Bik\left[j'_\ell(ika) + i\eta'_\ell(ika)\right] - Aikj'_\ell(ika) = -\lambda A j_\ell(ika).$$

Substituting our expression for A found above from continuity, we get:

$$B\left[j'_\ell(ika) + i\eta'_\ell(ika)\right] - B\left(1 + i\frac{\eta_\ell(ika)}{j_\ell(ika)}\right)j'_\ell(ika) = -\frac{\lambda}{k}B\left(1 + i\frac{\eta_\ell(ika)}{j_\ell(ika)}\right)j_\ell(ika).$$

Simplifying, this gives:

$$ij_\ell(ika)\eta'_\ell(ika) - i\eta_\ell(ika)j'_\ell(ika) = \frac{i\lambda}{k}j_\ell(ika)\big[j'_\ell(ika) + i\eta_\ell(ika)\big].$$

We can now examine this in the limit when $k \to 0$ using the following limits:

$$\begin{cases} j_\ell(ika) \to \frac{(ika)^\ell}{(2\ell+1)!!} \\ j'_\ell(ika) \to \frac{\ell}{ika}j_\ell(ika) \\ \eta_\ell(ika) \to -\frac{(2\ell+1)!!}{(ika)^{\ell+1}} \\ \eta'_\ell(ika) \to -\frac{\ell+1}{ika}\eta_\ell(ika) \end{cases}$$

Hence, for the combination of the two, we get:

$$j_\ell(ika)\eta_\ell(ika) \to -\frac{1}{ika}\frac{1}{2\ell+1}.$$

Giving us:

$$-\frac{(\ell+1)}{ika}\left(-\frac{1}{ika}\right)\frac{1}{2\ell+1} - \frac{\ell}{ika}\left(-\frac{1}{ika}\right)\frac{1}{2\ell+1} = \frac{\lambda i}{k}\left(-\frac{1}{ika}\right)\frac{1}{2\ell+1}$$

and $(2\ell+1) = \lambda a$. Hence, when $\lambda a = 2\ell + 1$, a bound state appears at $E = 0$! There is a connection between the appearance of a bound state and the divergence of a scattering amplitude as $k \to 0$. In fact, if we think of the scattering amplitudes as functions of complex k, then bound states appear at the poles of the scattering amplitude, which is an alternate way to solve bound state problems!

4 Delta shell potential example

For the delta shell potential, we have:

$$f_\ell(k) = \frac{1}{k}\frac{ka\lambda a j_\ell^2(ka)}{1 - ika\lambda a j_\ell(ka)\big[j_\ell(ka) + i\eta_\ell(ka)\big]}.$$

The poles occur at:

$$1 = ika\lambda a j_\ell(ka)\big[j_\ell(ka) + i\eta_\ell(ka)\big]$$

where we need to substitute $k \to iK$ and solve for poles. This will give the formula for the bound state energies:

$$1 = -Ka\lambda a j_\ell(iKa)\big[j_\ell(iKa) + i\eta_\ell(iKa)\big].$$

The general rule is that a positive scattering length implies a weakly bound state and strong scattering near $k \to 0$ while a negative scattering length implies a pre-bound state and strong scattering near $k \to 0$.

5 Experimentally tuning Feshbach resonances

In atomic physics, one sweeps the magnetic field across the resonance and form bound state molecules (weakly bound), which can then be studied or made into more deeply bound objects.

We can show this condition holds by verifying the Wronskian:

$$j_\ell \eta_\ell' - \eta_\ell j_\ell' = \frac{1}{z^2}$$

Proof. Begin with the equations

$$z^2\frac{d^2}{dz^2}\eta_\ell + 2z\frac{d}{dz}\eta_\ell + \big(z^2 - \ell(\ell+1)\big)\,\eta_\ell = 0$$
$$z^2\frac{d^2}{dz^2}j_\ell + 2z\frac{d}{dz}j_\ell + \big(z^2 - \ell(\ell+1)\big)\,j_\ell = 0.$$

Now multiply the first equation by j_ℓ and the second equation by η_ℓ, subtracting the results to get:

$$z^2\left[j_\ell\eta_\ell'' - \eta_\ell j_\ell''\right] + 2z\left[j_\ell\eta_\ell' - \eta_\ell j_\ell'\right] = 0$$

$$z^2\frac{d}{dz}\left[j_\ell\eta_\ell' - \eta_\ell j_\ell'\right] + 2z\left[j_\ell\eta_\ell' - \eta_\ell j_\ell'\right] = 0$$

$$-\frac{2}{z}dz = d\ln[j_\ell\eta_\ell' - \eta_\ell j_\ell']$$

$$-2\ln z + c = \ln[j_\ell\eta_\ell' - \eta_\ell j_\ell']$$

which implies:

$$\frac{c}{z^2} = j_\ell \eta'_\ell - \eta_\ell j'_\ell.$$

Checking the limit as $z \to 0$, gives $C = 1$, so:

$$j_\ell \eta'_\ell - \eta_\ell j'_\ell = \frac{1}{z^2}$$

or:

$$-\frac{1}{(ka)^2} = \frac{\lambda}{k} j_\ell(ika) \left[j_\ell(ika) + i\eta_\ell(ika) \right].$$

$$1 = -\lambda aka j_\ell(ika) \left[j_\ell(ika) + i\eta_\ell(ika) \right].$$

Take $k \to i\kappa$ in the scattering amplitude to find the same equation for the bound state condition. □