Phys 506 lecture 26: Interaction representation

1 Review of the time-ordered product

Last time we developed an expansion for $\hat{U}(t,t_0)$ in powers of $\hat{H}(t)$:

$$
\begin{aligned}
|\psi(t)\rangle &= \hat{U}\left(t,t_0\right)|\psi\left(t_0\right)\rangle \\
\hat{U}\left(t,t_0\right) &= T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\,\hat{H}\left(t'\right)\right] \\
&= \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_{t_0}^{t} dt_1\cdots\int_{t_0}^{t} dt_n\, T\left(\hat{H}\left(t_1\right)\cdots H\left(t_n\right)\right) \\
&= \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\cdots\int_{t_0}^{t_{n-1}} dt_n\,\hat{H}\left(t_1\right)\hat{H}\left(t_2\right)\cdots H\left(t_n\right)
\end{aligned}
$$

Note also that the equation of motion for $\hat{U}$ is

$$
i\hbar\frac{\partial}{\partial t}\hat{U}\left(t,t_0\right) = \hat{H}(t)\hat{U}(t,t_0)
$$

which follows from the last form for $\hat{U}$. Now, note that t only appears in the upper limit of the last integral, which allows us to take the derivative with respect to t as follows:

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\hat{U}\left(t,t_0\right) &= \sum_{n=0}^{\infty} -\left(\frac{i}{\hbar}\right)^n i\hbar\frac{\partial}{\partial t}\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\cdots\int_{t_0}^{t_{n-1}} dt_n\hat{H}\left(t_1\right)\hat{H}\left(t_2\right)\cdots\hat{H}\left(t_n\right) \\
&= \frac{-i}{\hbar}\left(i\hbar\hat{H}(t)\right)\sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{t_0}^{t} dt_1\int_{t_0}^{t_2} dt_2\cdots\int_{t_0}^{t_{n-1}} dt_n\hat{H}\left(t_1\right)\cdots H\left(t_n\right) \\
&= \hat{H}(t)\hat{U}\left(t_1\left(t_0\right)\right.\,.
\end{aligned}
$$

But in many cases, the Hamiltonian separates into a time independent piece and a small time-dependent piece as $\hat{H} = \hat{H}_0 + \hat{V}(t)$. In those cases, we want an expansion in $\hat{V}$ not $\hat{H}$.

2 Pictures for time evolution

We start by looking at different pictures for quantum mechanizes.

We are all familiar with the Schrodinger representation.

$$
\hat{H}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle \qquad \text{assume } \hat{H} \text{ independent of time here.}
$$

Now consider the expectation value of an operator $\hat{A}$ with no time dependence, where the expectation value can have time dependence from the dime dependence of the states:

$$A(t) = \langle\psi(t)|\hat{A}|\psi(t)\rangle.$$

Then, taking the derivative gives us

$$\begin{aligned}
\frac{d}{dt}A(t) &= \left(\frac{\partial}{\partial t}\langle\psi(t)|\right)\hat{A}|\psi(t)\rangle + \langle\psi(t)|\hat{A}\frac{\partial}{\partial t}|\psi(t)\rangle \\
&= \frac{i}{\hbar}\langle\psi(t)|\hat{H}\hat{A}|\psi(t)\rangle - \frac{i}{\hbar}\langle\psi(t)|\hat{A}\hat{H}|\psi(t)\rangle \\
&= -\frac{i}{\hbar}\langle\psi(t)|[\hat{A},\hat{H}]|\psi(t)\rangle \\
i\hbar\frac{d}{dt}A(t) &= \langle\psi(t)|[\hat{A},\hat{H}]|\psi(t)\rangle.
\end{aligned}$$

All time dependence comes from the wave functions in the Schrodinger representation when $\frac{\partial H}{\partial t} = 0$.

Heisenberg representation

Write $|\psi_s(t)\rangle = \hat{U}(t)\,|\psi_H\rangle$ with $\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$ Then define

$$\hat{A}_H(t) = \hat{U}^\dagger(t)\hat{A}\hat{U}(t).$$

Then

$$\begin{aligned}
A(t) &= \langle\psi_s(t)|\,\hat{A}\,|\psi_s(t)\rangle \\
&= \langle\psi_H|\,\hat{U}^\dagger(t)\hat{A}\hat{U}(t)\,|\psi_H\rangle \\
&= \langle\psi_H|\,\hat{A}_H(t)\,|\psi_H\rangle\,,
\end{aligned}$$

and all time dependence is in the operators now. We find that

$$\begin{aligned}
&i\hbar\frac{d}{\partial t}\hat{A}_H(t) = i\hbar\frac{\partial}{\partial t}\hat{U}^\dagger\hat{A}\hat{U} + \hat{U}^\dagger\hat{A}i\hbar\frac{\partial}{\partial t}\hat{U} \\
&= -\hat{U}^\dagger\hat{H}\hat{A}\hat{U} + \hat{U}^\dagger\hat{A}H\hat{U} \\
&= \hat{U}^\dagger[\hat{A},\hat{H}]U \\
&= \left[\hat{A}_H(t),\hat{H}\right] \quad \text{since } \left[\hat{U},H\right] = 0 \text{ for time-independent } \hat{H}.
\end{aligned}$$

Summarizing, we have that

$$\boxed{i\hbar\frac{d}{dt}\hat{A}_H(t) = \left[\hat{A}_H(t),\hat{H}\right]}$$

which is called the equation of motion. All time dependence is now in the operators, which evolve like Poisson brackets in classical mechanics.

3 Pictures for time-dependent Hamiltonians

If $\hat{H}$ depends on time, $\hat{H}_s(t)$, we proceed similarly

$$\begin{aligned}|\psi_s(t)\rangle &= \hat{U}\left(t,t_0\right)|\psi_H\left(t_0\right)\rangle\\ \hat{A}_H(t) &= \hat{U}^\dagger\left(t,t_0\right)\ \hat{A}_s(t)\ \hat{U}\left(t,t_0\right)\\ \hat{H}_H(t) &= \hat{U}^\dagger\left(t,t_0\right)\ \hat{H}_s(t)\ \hat{U}\left(t,t_0\right)\end{aligned}$$

$$i\hbar\frac{\partial}{\partial t}\hat{U}\left(t,t_0\right) = \hat{H}_s(t)\hat{U}\left(t,t_0\right),$$

so

$$\begin{aligned}\frac{\partial}{\partial t}\hat{A}_H(t) =& i\hbar\frac{\partial}{\partial t}\hat{U}^\dagger\left(t,t_0\right)\hat{A}_s(t)U(t,t_0)\\ &+ i\hbar\hat{U}^\dagger(t,t_0)\frac{\partial}{\partial t}\hat{A}_s(t)\hat{U}(t,t_0) + i\hbar\hat{U}^\dagger(t,t_0)\hat{A}_s(t)\frac{\partial}{\partial t}\hat{U}(t,t)\\ =& -\hat{U}^\dagger\left(t,t_0\right)\hat{H}_s(t)\hat{A}_s(t)\hat{U}\left(t,t_0\right)\\ &+ i\hbar\hat{U}^\dagger(t,t_0)\frac{\partial}{\partial t}\hat{A}_s(t)\hat{U}(t,t_0) + \hat{U}^\dagger\left(t,t_0\right)\hat{A}_s(t)\hat{H}_s(t)\hat{U}(t,t_0).\end{aligned}$$

but

$$\hat{U}^\dagger\hat{H}_s\hat{A}_s\hat{U} = \hat{U}^\dagger\hat{H}_s\hat{U}^\dagger\hat{U}^\dagger\hat{A}_s\hat{U} = \hat{H}_H(t)\hat{A}_H(t),$$

so

$$\boxed{i\hbar\frac{\partial}{\partial t}\hat{A}_H(t) = \left[\hat{A}_H(t),\hat{H}_H(t)\right] + i\hbar\frac{\partial\hat{A}_H}{\partial t}(t)}.$$

The last term is $\hat{U}^\dagger\left(t,t_0\right)\frac{\partial A_s(t)}{\partial t}\hat{U}\left(t,t_0\right) = i\hbar\frac{\partial\hat{A}_H(t)}{\partial t}$.

4 Interaction representation

The interaction representation is halfway between Schrodinger and Heisenberg. We have the same break-up into a time-independent and a small time-dependent piece:

$$\hat{H}(t) = \hat{H} + \hat{V}(t).$$

So we have

$$\hat{H}_0|n\rangle = E_n^0|n\rangle.$$

Define

$$\begin{aligned}|\psi_I(t)\rangle &= e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\,|\psi_s(t)\rangle\\ \hat{A}_I(t) &= e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}A_s(t)e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}\\ \hat{V}_I(t) &= e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\hat{V}_s(t)e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}.\end{aligned}$$

Note that $H_{0I}(t) = H_{0s}(t) = H_0(t)$ since

$$\left[\hat{H}_0, e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\right] = 0.$$

So, we find that

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\left|\psi_I(t)\right\rangle &= i\hbar\frac{d}{dt}\left[e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\left|\psi_s(t)\right\rangle\right] \\ &= -\hat{H}_0 e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\left|\psi_s(t)\right\rangle + e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\hat{H}_s(t)\left|\psi_s(t)\right\rangle \\ &= e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\left(\underbrace{-\hat{H}_0+\hat{H}}_{0}+\hat{V}_s(t)\right)\left|\psi_s(t)\right\rangle \\ &= e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\hat{V}_s(t)e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\left|\psi_s(t)\right\rangle \\ &= \hat{V}_I(t)\left|\psi_I(t)\right\rangle. \end{aligned}$$

Hence, if $\hat{U}_I\left(t_1 t_0\right)\left|\psi_I\left(t_0\right)\right\rangle = \left|\psi_I(t)\right\rangle$, then we have

$$i\hbar\frac{\partial}{\partial t}\hat{U}_I\left(t,t_0\right) = \hat{V}_I(t)\hat{U}_I\left(t,t_0\right).$$

$$\boxed{\begin{aligned} &\Rightarrow \hat{U}_I\left(t,t_0\right) = T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{V}_I\left(t'\right)dt'\right] \\ &\quad \hat{U}_I\left(t,t_0\right) = \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n\frac{1}{n!}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\cdots\int_{t_0}^{t}dt_n T\left[\hat{V}_I\left(t_1\right)\hat{V}_I\left(t_2\right)\cdots\hat{V}_I\left(t_n\right)\right] \end{aligned}}$$

This is called the Dyson expansion for the evolution operator.

Note that

$$\begin{aligned} \left|\psi_I(t)\right\rangle &= e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\left|\psi_s(t)\right\rangle = e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}\hat{U}_s\left(t,t_0\right)\left|\psi_s\left(t_0\right)\right\rangle \\ &= \hat{U}_I\left(t,\,t_0\right)\left|\psi_I\left(t_0\right)\right\rangle \end{aligned}$$

but $\left|\psi_I\left(t_0\right)\right\rangle = \left|\psi_S\left(t_0\right)\right\rangle = \left|\psi_H\left(t_0\right)\right\rangle$ since $U\left(t_0,t_0\right)=1$ in all pictures.

So $\boxed{U_s\left(t,t_0\right) = e^{-\frac{i}{\hbar}\hat{H}_0(t-t)}T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}dt'\hat{V}_I\left(t'\right)\right]}$

This is the desired expansion of the evolution operator in a power series in $\hat{V}$ and it will be used for time dependent perturbation theory in a few lectures.

We can derive this result directly though. Recall $i\hbar\frac{\partial}{\partial t}Te^{-\frac{i}{\hbar}\int_{t_0}^{t}dt'\hat{A}(t')} = \hat{A}(t)Te^{-\frac{i}{\hbar}\int_{t_0}^{t}dt'\hat{A}(t')}$ so

$$\begin{aligned} &i\hbar\frac{d}{dt}\left[e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}dt'\;\hat{V}_I\left(t'\right)\right]\right] \\ &= \hat{H}_0 e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{V}_I\left(t'\right)dt'\right] + e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}\hat{V}_I(t)T\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{V}_I\left(t'\right)dt'\right]. \end{aligned}$$

But $e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}\hat{V}_I(t) = \hat{V}_s(t)e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}$ so

$$= \left[\underbrace{\hat{H}_0+\hat{V}_s(t)}_{\hat{H}(t)}\right]e^{-\frac{i}{\hbar}\hat{H}_0(t-t)}Te^{-\frac{i}{\hbar}\int_{t_0}^{t}dt'\hat{V}_I(t)}.$$

So $i\hbar\frac{\partial}{\partial t}O_p(t,t_0) = \hat{H}(t)\;O_p(t,t_0)$ and $O_p\left(t_0,t_0\right)=\mathbb{I}$, which implies that $O_p(t,t_0) = \hat{U}_s\left(t,t_0\right)$, so we have proved the result directly.