Phys 506 lecture 27: Cyclotron resonance

1 Precession of a spin

An example of an exact time-ordered product.

When a spin is placed in a constant magnetic field, it precesses as shown below.

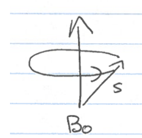

Figure 1: Precession involves a rotating object, whose axis makes an angle to the vertical, and over time, the axis revolves in a circle about the vertical, with the tail fixed on the vertical, and the tip tracing out a circle in the horizontal plane.

The magnetic moment of a spin, with angular momentum $\mathbf{S}$ is $\mu = \gamma \mathbf{S}$, where we are using the Gottfried normalisation for $\mathbf{S}$ and the factor $\gamma = \frac{eg\hbar}{2mc}$ is called the gyromagnetic ration with the Lande g factor g. If we study the Hamiltonian in a magnetic field $\mathbf{B}_0$ along the $\hat{z}$ direction, we get:

$$\hat{H}_0 = -\mu \cdot \mathbf{B}_0 = -\gamma B_0 \hat{S}_z = -\hbar\Omega\hat{S}_z$$

where $\Omega = \frac{\gamma B_0}{\hbar}$ is called the Larmor frequency. If our states had total spin s and z component m, then:

$$\hat{H}_0|s,m\rangle = -\hbar\Omega m|s,m\rangle.$$

The time dependence is:

$$|s,m;t\rangle = e^{-\frac{i}{\hbar}\hat{H}_0 t}|s,m\rangle = e^{i\Omega m t}|s,m;t=0\rangle.$$

Hence, the states rotate with a frequency Ωm. This rotation is called precession (harmonics of the Larmor frequency). In a magnetic resonance experiment, we add an additional *small* perpendicular field that rotates around the z-axis.

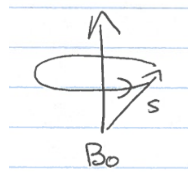

Figure 2: Precession involves a rotating object, whose axis makes an angle to the vertical, and over time, the axis revolves in a circle about the vertical, with the tail fixed on the vertical, and the tip tracing out a circle in the horizontal plane. The added magnetic field is also in the horizontal plane.

2 Magnetic resonance

The second field is small, but its rotation frequency can be adjusted externally. Interesting things happen when the second field rotates at the same rate as the precession of the spin. Then, in the spin's rest frame, it sees a static perpendicular magnetic field, which the spin will precess about. If we wait long enough, the spin will flip. Let's examine this mathematically:

$$\mathbf{B}(t) = B_0\mathbf{e}_z + B_1(\mathbf{e}_x \cos\omega t - \mathbf{e}_y \sin\omega t)$$

$$\hat{H}(t) = -\hbar\mu \cdot \mathbf{B} = -\hbar\Omega\hat{S}_z - \gamma B_1(\hat{S}_x \cos\omega t - \hat{S}_y \sin\omega t).$$

Now consider the operator that transforms us into the rotating frame of the transverse magnetic field $\mathbf{B}$,

$$\hat{D}(\omega t, \hat{z}) = e^{-i\omega t\hat{S}_z}$$

We find that

$$\hat{D}^\dagger(\omega t, \hat{z})\hat{S}_z\hat{D}(\omega t, \hat{z}) = e^{i\omega t\hat{S}_z}\hat{S}_z e^{-i\omega t\hat{S}_z} = \hat{S}_z$$

and:

$$\hat{D}^\dagger(\omega t, \hat{z})\hat{S}_x\hat{D}(\omega t, \hat{z}) = e^{i\omega t\hat{S}_z}\hat{S}_x e^{-i\omega t\hat{S}_z} = f_x(t)$$

The time derivative can be found to be:

$$\frac{df_x}{dt} = i\omega e^{i\omega t\hat{S}_z}[\hat{S}_z, \hat{S}_x]e^{-i\omega t\hat{S}_z}.$$

Using the commutator $[\hat{S}_z, \hat{S}_x] = i\hat{S}_y$ (Gottfried normalization), we find:

$$\frac{df_x}{dt} = -\omega f_y,$$

where $f_y = e^{i\omega t\hat{S}_z}\hat{S}_y e^{-i\omega t\hat{S}_z}$, which again has a time derivative:

$$\frac{df_y}{dt} = i\omega e^{i\omega t\hat{S}_z}[\hat{S}_z, \hat{S}_y]e^{-i\omega t\hat{S}_z} = \omega f_x.$$

Thus, we get the system of differential equations:

$$\begin{cases} \frac{d^2 f_x}{dt^2} = -\omega^2 f_x \\ \frac{d^2 f_y}{dt^2} = -\omega^2 f_y \end{cases}$$

The solutions are:

$$\begin{cases} f_x(t) = \cos\omega t\, f_x(0) + \frac{1}{\omega}\sin\omega t\, f_x(0) \\ f_y(t) = \cos\omega t\, f_y(0) + \frac{1}{\omega}\sin\omega t\, f_y(0) \end{cases}$$

with initial conditions:

$$\begin{cases} f_x(0) = \hat{S}_x \\ f_y(0) = \hat{S}_y \\ \dot{f}_x(t) = -\omega\hat{S}_y \\ \dot{f}_y(t) = \omega\hat{S}_x \end{cases}$$

Therefore:

$$\begin{cases} \hat{D}^\dagger(\omega t, \hat{z})\hat{S}_x\hat{D}(\omega t, \hat{z}) = \cos\omega t\,\hat{S}_x - \sin\omega t\,\hat{S}_y \\ \hat{D}^\dagger(\omega t, \hat{z})\hat{S}_y\hat{D}(\omega t, \hat{z}) = \cos\omega t\,\hat{S}_y + \sin\omega t\,\hat{S}_x \end{cases}$$

Hence, we can write:

$$\boxed{\hat{H}(t) = \hat{D}^\dagger(\omega t, \hat{z})\left[-\hbar\Omega\hat{S}_z - \gamma B_1\hat{S}_x\right]\hat{D}(\omega t, \hat{z}).}$$

The Hamiltonian is a unitary transformation of a time-independent Hamiltonian!

3 Going to the rotating frame

To work in the rotating frame, we write:

$$|\psi(t)\rangle = \hat{D}(\omega t, \hat{z})|\psi_R(t)\rangle$$

where the R indicates a rotating frame. Substituting into the Schrödinger equation and using the property $\hat{D}^\dagger\hat{D} = 1$, we get:

$$\hat{H}(t)|\psi(t)\rangle = \hat{D}^\dagger(\omega t, \hat{z})\left[-\hbar\Omega\hat{S}_z - \gamma B_1\hat{S}_x\right]|\psi_R(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi_R(t)\rangle.$$

Multiply through by $\hat{D}(\omega t, \hat{z})$:

$$[-\hbar\Omega\hat{S}_z - \gamma B_1\hat{S}_x]|\psi_R(t)\rangle = i\hbar\hat{D}(\omega t, \hat{z})\frac{\partial}{\partial t}\hat{D}^\dagger(\omega t, \hat{z})|\psi_R(t)\rangle = -\hbar\omega\hat{S}_z|\psi_R(t)\rangle + i\hbar\frac{\partial}{\partial t}|\psi_R(t)\rangle.$$

Rearranging we get:

$$[(\hbar\omega - \hbar\Omega)\hat{S}_z - \gamma B_1\hat{S}_x]|\psi_R(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi_R(t)\rangle,$$

where the Hamiltonian is now time independent and hence easier to solve. Namely, the time-evolved state is given by:

$$|\psi_R(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi_R(0)\rangle = e^{-\frac{i}{\hbar}t\left[(\hbar\omega-\hbar\Omega)\hat{S}_z-\gamma B_1\hat{S}_x\right]}|\psi_R(0)\rangle$$

So $|\psi(t)\rangle = \hat{D}^\dagger(\omega t,\hat{z})|\psi_R(t)\rangle$ or

$$\boxed{|\psi(t)\rangle = e^{-i\omega t\hat{S}_z}e^{-it\left[(\omega-\Omega)\hat{S}_z-\frac{\gamma B_1}{\hbar}\hat{S}_x\right]}|\psi_R(0)\rangle}$$

Suppose the system is initially in a pure state $|j,m\rangle$ at $t=0$. What is the probability of being in another state at time t?

$$P(t)_{j'm'\leftarrow jm} = \left|\langle j',m'|e^{-i\omega t\hat{S}_z}e^{-it\left[(\omega-\Omega)\hat{S}_z-\frac{\gamma B_1}{\hbar}\hat{S}_x\right]}|j,m\rangle\right|^2.$$

Since $\hat{S}^2$ commutes with $e^{i(\alpha\hat{S}_z+\beta\hat{S}_x)}$ and $|e^{i\omega tm'}|^2=1$, this simplifies to:

$$P(t)_{j'm'\leftarrow jm} = \delta_{jj'}\left|\langle m'|e^{-it\left[(\omega-\Omega)\hat{S}_z-\frac{\gamma B_1}{\hbar}\hat{S}_x\right]}|m\rangle\right|^2.$$

Calculating this result explicitly is complicated.

4 Spin one-half example

For a spin-$\frac{1}{2}$ system, the spin operators are:

$$\hat{S}_z = \frac{1}{2}\sigma_z = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \hat{S}_x = \frac{1}{2}\sigma_x = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Derived earlier in class, we found:

$$e^{i\mathbf{u}\cdot\sigma} = \cos|\mathbf{u}| + i\sin|\mathbf{u}|\frac{\mathbf{u}\cdot\sigma}{|\mathbf{u}|}$$

Hence, we find that:

$$e^{-it\left[(\omega-\Omega)\hat{S}_z-\frac{\gamma B_1}{\hbar}\hat{S}_x\right]} = e^{-i\mathbf{u}\cdot\sigma}$$

where

$$|\mathbf{u}| = \frac{t}{2}\sqrt{(\omega-\Omega)^2+\left(\frac{\gamma B_1}{\hbar}\right)^2} = \Delta\frac{t}{2}$$

and where we defined $\Delta = \sqrt{(\omega-\Omega)^2+\left(\frac{\gamma B_1}{\hbar}\right)^2}$. Thus, using the identity derived previsouly in class that we mentioned above, the exponential becomes:

$$e^{-it\left[(\omega-\Omega)\hat{S}_z-\frac{\gamma B_1}{\hbar}\hat{S}_x\right]} = \cos\left(\frac{\Delta t}{2}\right)\mathbb{I} - i\sin\left(\frac{\Delta t}{2}\right)\left[\frac{(\omega-\Omega)}{\Delta}\sigma_z - \frac{\gamma B_1}{\hbar\Delta}\sigma_x\right].$$

The probability amplitudes for the states are:

$$\left|\left\langle\frac{1}{2},-\frac{1}{2}\right|\cos\left(\frac{\Delta t}{2}\right)\mathbb{I}-i\sin\left(\frac{\Delta t}{2}\right)\left[\frac{(\omega-\Omega)}{\Delta}\sigma_z-\frac{\gamma B_1}{\hbar\Delta}\sigma_x\right]\left|\frac{1}{2},\frac{1}{2}\right\rangle\right|^2$$
$$=\sin^2\left(\frac{\Delta t}{2}\right)\frac{\gamma^2B_1^2}{\hbar^2\Delta^2}=P_{-\frac{1}{2}\leftarrow\frac{1}{2}}(t)$$

Similarly, the probability for remaining in the initial state is:

$$P_{\frac{1}{2}\to\frac{1}{2}}(t)=\left|\cos\left(\frac{\Delta t}{2}\right)-i\sin\left(\frac{\Delta t}{2}\right)\frac{(\omega-\Omega)}{\Delta}\right|^2=\cos^2\left(\frac{\Delta t}{2}\right)+\sin^2\left(\frac{\Delta t}{2}\right)\frac{(\omega-\Omega)^2}{\Delta^2}$$
$$=1-\sin^2\left(\frac{\Delta t}{2}\right)\frac{\gamma^2B_1^2}{\hbar^2\Delta^2}.$$

We can see that our results are consistent as:

$$P_{\frac{1}{2}\to\frac{1}{2}}(t)+P_{-\frac{1}{2}\leftarrow\frac{1}{2}}(t)=1,$$

should hold by probability conservation.

Note that higher spin cases can be worked out with the exponential disentangling identity.