Phys 506 lecture 28: An exact time-ordered product

1 Driven simple harmonic oscillator

We want to solve the time-dependent Schrödinger equation for the full Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}(t)$ where

$$\hat{H}_0 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \quad \text{and} \quad \hat{V}(t) = f(t)\hat{a}^\dagger + f^*(t)\hat{a}$$

Since $\hat{a} + \hat{a}^\dagger \propto \hat{x}$, $\hat{V}(t)$ acts like a driving force moving x as a function of time. Recall that $[\hat{a}, \hat{a}^\dagger] = 1$. Then, define

$$\hat{A} = \hat{a} + \frac{f(t)}{\hbar\omega}$$

One can easily verify that

$$[\hat{A}, \hat{A}^\dagger] = [\hat{a}, \hat{a}^\dagger] = 1$$

Then,

$$\hbar\omega\left(\hat{A}^\dagger\hat{A} + \frac{1}{2}\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\frac{f(t)}{\hbar\omega} + \hat{a}\frac{f^*(t)}{\hbar\omega} + \frac{|f(t)|^2}{(\hbar\omega)^2} + \frac{1}{2}\right) = \hat{H}(t) + \frac{|f(t)|^2}{\hbar\omega}$$

So,

$$\hat{H}(t) = \hbar\omega\left(\hat{A}^\dagger\hat{A} + \frac{1}{2}\right) - \frac{|f(t)|^2}{\hbar\omega}$$

We define the following:

$$\hat{A}\left|0\right\rangle = 0, \quad \left|n\right\rangle = \frac{(\hat{A}^\dagger)^n}{\sqrt{n!}}\left|0\right\rangle$$

Then,

$$\hat{H}\left|n\right\rangle = E_n(t)\left|n\right\rangle = \left[\hbar\omega\left(n + \frac{1}{2}\right) - \frac{|f(t)|^2}{\hbar\omega}\right]\left|n\right\rangle$$

Expand $\left|\psi(t)\right\rangle = \sum_n c_n(t)\left|n\right\rangle$.

$$i\hbar\,\partial_t\left|\psi(t)\right\rangle = \sum_n i\hbar\left[\partial_t c_n(t)\left|n\right\rangle + c_n(t)\partial_t\left|n\right\rangle\right] = \hat{H}\left|\psi(t)\right\rangle = \sum_n c_n(t)E_n(t)\left|n\right\rangle.$$

Multiply by $\langle m|$ to get

$$i\hbar\,\partial_t c_m(t) + i\hbar\sum_n c_n(t)\left\langle m|\partial_t|n\right\rangle = E_m(t)c_m(t)$$

So we need to calculate

$$\partial_t |n\rangle = \partial_t \frac{(\hat{A}^\dagger)^n}{\sqrt{n!}} |0\rangle = \partial_t \left(\frac{\left(\hat{a}^\dagger + \frac{f^*(t)}{\hbar\omega}\right)^n}{\sqrt{n!}} \right) |0\rangle$$

$$= n \frac{\left(\hat{a}^\dagger + \frac{f^*(t)}{\hbar\omega}\right)^{n-1}}{\sqrt{n!}} \frac{df^*(t)}{dt} \frac{1}{\hbar\omega} |0\rangle + \frac{(\hat{A}^\dagger)^n}{\sqrt{n!}} \partial_t |0\rangle$$

$$= \frac{\sqrt{n}}{\hbar\omega} \frac{df^*(t)}{dt} |n-1\rangle + \frac{(\hat{A}^\dagger)^n}{\sqrt{n!}} \partial_t |0\rangle$$

But $\hat{A}|0\rangle = 0$ implies that $\partial_t \left(\hat{a} + \frac{f(t)}{\hbar\omega}\right)|0\rangle = 0$, which further implies that

$$\frac{df}{dt}\frac{1}{\hbar\omega}|0\rangle + \left(\hat{a} + \frac{f(t)}{\hbar\omega}\right)\partial_t|0\rangle = 0.$$

Multiply by $\langle m|$ to get

$$\frac{df}{dt}\frac{1}{\hbar\omega}\delta_{m0} + \left\langle m \middle| \hat{A}\partial_t \middle| 0 \right\rangle = 0$$

$$\implies \sqrt{m+1}\,\langle m+1|\partial_t|0\rangle = -\frac{df}{dt}\frac{1}{\hbar\omega}\delta_{m0}$$

so,

$$\langle m|\partial_t|0\rangle = -\frac{df}{dt}\frac{1}{\hbar\omega}\delta_{m1}$$

$$\implies \partial_t|n\rangle = \frac{\sqrt{n}}{\hbar\omega}\frac{df^*}{dt}|n-1\rangle - \frac{\sqrt{n+1}}{\hbar\omega}\frac{df}{dt}|n+1\rangle.$$

Substituting back into the Schrödinger equation,

$$i\hbar\,\partial_t c_m(t) + i\hbar\frac{\sqrt{m+1}}{\hbar\omega}\frac{df^*}{dt}c_{m+1}(t) - \frac{i\hbar\sqrt{m}}{\hbar\omega}\frac{df}{dt}c_{m-1}(t) = E_m(t)c_m(t)$$

So we have,

$$\boxed{\frac{dc_m(t)}{dt} = -\frac{i}{\hbar}E_m(t)c_m(t) - \frac{\sqrt{m+1}}{\hbar\omega}\frac{df^*}{dt}c_{m+1}(t) + \frac{\sqrt{m}}{\hbar\omega}\frac{df}{dt}c_{m-1}(t)}$$

This equation is a complicated coupled linear differential equation with no obvious solution when $f \neq 0$. So, this attempt at a solution does not work.

2 Interaction representation

Let's examine instead with the interaction representation picture.

$$|\psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_0 t}\,T\exp\left(-\frac{i}{\hbar}\int_0^t dt'\hat{V}_I(t')\right)|\psi_S(0)\rangle$$

where

$$\hat{V}_I(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{V}(t)e^{-\frac{i}{\hbar}\hat{H}_0 t}.$$

Note that V_I is different from V, Not recognizing this is a common mistake and will lead to errors. In our case, we have

$$\hat{V}_I(t) = e^{\frac{i}{\hbar}\hbar\omega\hat{a}^\dagger\hat{a}t}[f\hat{a}^\dagger + f^*\hat{a}]e^{-i\omega\hat{a}^\dagger\hat{a}t}.$$

But $e^{i\lambda\hat{a}^\dagger\hat{a}}\hat{a}^\dagger e^{-i\lambda\hat{a}^\dagger\hat{a}} \equiv g(\lambda)$ so

$$ie^{i\lambda\hat{a}^\dagger\hat{a}}[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger]e^{-i\lambda\hat{a}^\dagger\hat{a}} = \frac{dg(\lambda)}{d\lambda}$$

$$ig(\lambda) = \frac{dg(\lambda)}{d\lambda} \implies g(\lambda) = g(0)e^{i\lambda}.$$

So,

$$\boxed{\hat{V}_I(t) = f(t)e^{i\omega t}\hat{a}^\dagger + f^*(t)e^{-i\omega t}\hat{a}}.$$

It could also be evaluated in our standard way to evaluate Hadamards as before, yielding the same result. It is useful to note that

$$\begin{aligned}[\hat{V}_I(t), \hat{V}_I(t')] &= f(t)f^*(t')e^{i\omega t}e^{-i\omega t'}[\hat{a}^\dagger, \hat{a}] + f^*(t)f(t')e^{-i\omega t}e^{i\omega t'}[\hat{a}, \hat{a}^\dagger] \\ &= -2i\,\mathrm{Im}\left(f(t)f^*(t')e^{i\omega(t-t')}\right),\end{aligned}$$

which is just a number. So, $[\hat{V}_I(t), \hat{V}_I(t')]$ commutes with all operators.

To be concrete, choose $f(t) = Ce^{i\Omega t}$ with $C \in \mathbb{R}$. Then,

$$\hat{V}_I(t) = C(e^{i(\omega+\Omega)t}\hat{a}^\dagger + e^{-i(\omega+\Omega)t}\hat{a})$$

and

$$[\hat{V}_I(t), \hat{V}_I(t')] = -2iC^2\,\mathrm{Im}(e^{i\Omega}(t-t')e^{i\omega(t-t')}) = -2iC^2\sin((\omega+\Omega)(t-t')).$$

Gottfried says to consider

$$\hat{W}(t) = \int_0^t \hat{V}_I(t')\,dt' = \frac{C}{i(\omega+\Omega)}(e^{i(\Omega+\omega)t} - 1)\hat{a}^\dagger - \frac{C}{i(\omega+\Omega)}(e^{-i(\Omega+\omega)t} - 1)\hat{a}$$

Then,

$$\begin{aligned}[\hat{W}(t), \hat{V}_I(t)] &= \frac{C^2}{i(\omega+\Omega)}(1 - e^{-i(\omega+\Omega)t})(-1) - \frac{C^2}{i(\omega+\Omega)}(1 - e^{i(\omega+\Omega)t}) \\ &= -\frac{2C^2}{i(\omega+\Omega)}(1 - \cos(\omega+\Omega)t),\end{aligned}$$

which is just a number again. Now make the unitary transformation $|\psi_I(t)\rangle = e^{-\frac{i}{\hbar}\hat{W}(t)}|\psi_{II}(t)\rangle$. But

$$\left[i\hbar\,\partial_t - \hat{V}_I(t)\right]|\psi_I(t)\rangle = 0$$

So we have

$$e^{\frac{i}{\hbar}\hat{W}(t)}\left[i\hbar\,\partial_t - \hat{V}_I(t)\right]e^{-\frac{i}{\hbar}\hat{W}(t)}e^{\frac{i}{\hbar}\hat{W}(t)}|\psi_I(t)\rangle = 0$$

Hence,

$$e^{\frac{i}{\hbar}\hat{W}(t)}\left[i\hbar\,\partial_t - \hat{V}_I(t)\right]e^{-\frac{i}{\hbar}\hat{W}(t)}|\psi_{II}(t)\rangle = 0$$

Since $\hat{W}(t)$ does not commute with $\hat{V}_I(t)$, we cannot easily evaluate the derivative term. So let us expand the exponentials in a power series and then differentiate.

3 Computing the time-ordered product directly

First, operate the derivative on the RHS.

$$\left(1+\frac{i}{\hbar}\hat{W}+\frac{1}{2!}\left(\frac{i}{\hbar}\right)^2\hat{W}^2+\frac{1}{3!}\left(\frac{i}{\hbar}\right)^3\hat{W}^3+\cdots\right)\left[i\hbar\,\partial_t-\hat{V}_I\right]\left(1-\frac{i}{\hbar}\hat{W}+\frac{1}{2!}\left(\frac{-i}{\hbar}\right)^2\hat{W}^2+\frac{1}{3!}\left(\frac{-i}{\hbar}\right)^3\hat{W}^3+\cdots\right)$$

$$\begin{aligned}
&=\left(1+\frac{i}{\hbar}\hat{W}+\frac{1}{2!}\left(\frac{i}{\hbar}\right)^2\hat{W}^2+\frac{1}{3!}\left(\frac{i}{\hbar}\right)^3\hat{W}^3+\cdots\right)\Big\{i\hbar\,\partial_t-\hat{V}_I-i\hbar\left(i\hbar\dot{\hat{W}}+\hat{W}i\hbar\,\partial_t-\hat{V}_I\hat{W}\right)\\
&\qquad+\frac{1}{2!}\left(\frac{-i}{\hbar}\right)^2\left[i\hbar\dot{\hat{W}}\hat{W}+i\hbar\hat{W}\dot{\hat{W}}+\hat{W}^2i\hbar\,\partial_t-\hat{V}_I\hat{W}^2\right]\\
&\qquad+\frac{1}{3!}\left(\frac{-i}{\hbar}\right)^3\left[i\hbar\dot{\hat{W}}\hat{W}^2+i\hbar\hat{W}\dot{\hat{W}}\hat{W}+i\hbar\hat{W}^2\dot{\hat{W}}+\hat{W}^3i\hbar\,\partial_t-\hat{V}_I\hat{W}^3\right]+\cdots\Big\}
\end{aligned}$$

$$\begin{aligned}
&=i\hbar\,\partial_t-\hat{V}_I+\dot{\hat{W}}+\hat{W}\partial_t+\frac{i}{\hbar}\hat{V}_I\hat{W}-\hat{W}\partial_t-\frac{i}{\hbar}\hat{W}\hat{V}_I-\frac{1}{2}\frac{i}{\hbar}\left(\dot{\hat{W}}\hat{W}+\hat{W}\dot{\hat{W}}+\hat{W}^2\partial_t-\frac{i}{\hbar}\hat{V}_I\hat{W}^2\right)\\
&+\frac{i}{\hbar}\left(\hat{W}\dot{\hat{W}}+\hat{W}^2\partial_t+\frac{i}{\hbar}\hat{W}\hat{V}_I\hat{W}\right)-\frac{1}{2}\frac{i}{\hbar}\hat{W}^2\partial_t+\frac{1}{2}\frac{1}{\hbar^2}\hat{W}^2\hat{V}_I+\frac{1}{6}\left(\frac{i}{\hbar}\right)^2\left(\dot{\hat{W}}\hat{W}^2+\hat{W}\dot{\hat{W}}\hat{W}+\hat{W}^2\dot{\hat{W}}\right)\\
&\qquad-\frac{1}{2}\left(\frac{i}{\hbar}\right)^2\left(\hat{W}\dot{\hat{W}}\hat{W}+\hat{W}^2\dot{\hat{W}}\right)+\frac{1}{2}\left(\frac{i}{\hbar}\right)^2(\hat{W}^2\dot{\hat{W}})+\cdots
\end{aligned}$$

But $\dot{\hat{W}}=\hat{V}_I(t)$ so we get

$$\begin{aligned}
&=i\hbar\,\partial_t-\hat{V}_I+\hat{V}_I-\frac{i}{\hbar}[\hat{W},\hat{V}_I]-\frac{i}{2\hbar}\left(\hat{V}_I\hat{W}+\hat{W}\hat{V}_I-2\hat{W}\hat{V}_I\right)+\frac{1}{\hbar^2}\hat{V}_I\hat{W}^2-\frac{1}{\hbar^2}\hat{W}\hat{V}_I\hat{W}+\frac{1}{2\hbar^2}\hat{W}^2\hat{V}_I\\
&\qquad-\frac{1}{6\hbar^2}\left(\hat{V}_I\hat{W}^2+\hat{W}\hat{V}_I\hat{W}+\hat{W}^2\hat{V}_I\right)+\frac{1}{2\hbar^2}\left(\hat{W}\hat{V}_I\hat{W}+\hat{W}^2\hat{V}_I\right)-\frac{1}{2\hbar^2}\hat{W}^2\hat{V}_I+\cdots\\
&=i\hbar\,\partial_t-\frac{i}{2\hbar}[\hat{W},\hat{V}_I]+\frac{1}{\hbar^2}\left(\frac{1}{3}\hat{V}_I\hat{W}^2-\frac{2}{3}\hat{W}\hat{V}_I\hat{W}+\frac{1}{3}\hat{W}^2\hat{V}_I\right)+\cdots\\
&=i\hbar\,\partial_t-\frac{i}{2\hbar}[\hat{W},\hat{V}_I]+\frac{1}{3\hbar^2}\left(-[\hat{W},\hat{V}_I]\hat{W}+\hat{W}[\hat{W},\hat{V}_I]\right)+\cdots\\
&=i\hbar\,\partial_t-\frac{i}{2\hbar}[\hat{W},\hat{V}_I]+\frac{1}{3\hbar^2}[\hat{W},[\hat{W},\hat{V}_I]]+\cdots
\end{aligned}$$

All higher-order terms are multiple commutators, but $[\hat{W},\hat{V}_I]$ commutes with everything so

$$=i\hbar\,\partial_t-\frac{i}{2\hbar}[\hat{W},\hat{V}_I]=i\hbar\,\partial_t+\frac{C^2}{\hbar(\omega+\Omega)}(1-\cos{(\Omega+\omega)t}).$$

So

$$\begin{aligned}
&\left[i\hbar\,\partial_t+\frac{C^2}{\hbar(\omega+\Omega)}(1-\cos{(\Omega+\omega)t}\right]|\psi_{II}(t)\rangle=0\\
\implies&|\psi_{II}(t)\rangle=e^{\frac{i}{\hbar}\int_0^t\frac{C^2}{\hbar(\omega+\Omega)}(1-\cos{(\Omega+\omega)t}\,dt'}|\psi_{II}(0)\rangle\\
&=e^{\frac{iC^2}{\hbar^2(\omega+\Omega)}\left(t-\frac{\sin{(\omega+\Omega)t}}{\omega+\Omega}\right)}|\psi_{II}(0)\rangle\,.
\end{aligned}$$

Hence,

$$\begin{aligned} |\psi_I(t)\rangle &= e^{-\frac{i}{\hbar}\hat{W}(t)}\,|\psi_{II}(t)\rangle \\ &= \exp\left(-\frac{C}{\hbar(\omega+\Omega)}(e^{i(\omega+\Omega)t}-1)\hat{a}^\dagger + \frac{C}{\hbar(\omega+\Omega)}(e^{-i(\omega+\Omega)t}-1)\hat{a}\right) \\ &\times\ \exp\left(\frac{iC^2}{\hbar^2(\omega+\Omega)}(t-\frac{\sin{(\omega+\Omega)t}}{\omega+\Omega})\right)|\psi_{II}(0)\rangle \end{aligned}$$

and

$$|\psi_S(t)\rangle = e^{-i\omega t(\hat{a}^\dagger\hat{a}+\frac{1}{2})}\,|\psi_I(t)\rangle$$

To check, we could compute overlaps with states $\langle m|$ and verify the differential equation for $c_m(t)$ holds, but I won't do that.

In summary, when $[\hat{V}_I(t), \hat{V}_I(t')]$ is a number, the time-ordered product simplifies. In particular, we have

$$\boxed{\begin{aligned} T\exp(-\tfrac{i}{\hbar}\textstyle\int_0^t \hat{V}_I(t')\,dt') &= T\exp\left(-\tfrac{i}{\hbar}C\textstyle\int_0^t\left[e^{i(\omega+\Omega)t'}\hat{a}^\dagger + e^{-i(\omega+\Omega)t'}\hat{a}\right]\,dt'\right) \\ = \exp\left(-\tfrac{C}{\hbar(\omega+\Omega)}(e^{i(\omega+\Omega)t}-1)\hat{a}^\dagger + \tfrac{C}{\hbar(\omega+\Omega)}(e^{-i(\omega+\Omega)t}-1)\hat{a}\right)&\ \exp\left(\tfrac{iC^2}{\hbar^2(\omega+\Omega)}(t-\tfrac{\sin{(\omega+\Omega)t}}{\omega+\Omega})\right)\end{aligned}}$$

for this case.