Phys 506 lecture 3

In this lecture, we use the identities we just developed to determine the simple harmonic oscillator wavefunctions and other properties. The approach given here is a little different from what you will see in textbooks.

1 Factorizing the Hamiltonian

The SHO Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega_0^2 \hat{x}^2$$

Many textbooks postulate ladder operations as a "trick" solution, but if we think of factorizing, like with polynomials, we would try

$$\underbrace{\frac{1}{\sqrt{2m}}(\hat{p} + im\omega_0\hat{x})}_{\hat{A}^\dagger}\underbrace{\frac{1}{\sqrt{2m}}(\hat{p} - im\omega_0\hat{x})}_{\hat{A}}$$

for factorizing the sum of squares. You might next ask why not factor in the opposite order as $\hat{A}\hat{A}^\dagger$? We will answer the ordering question later.

Now, when we work out the product, because they are operators, we find an extra term from the commutator, or

$$\begin{aligned}\hat{A}^\dagger\hat{A} &= \frac{1}{2m}\left(\hat{p}^2 - im\omega_0 \underbrace{[\hat{p}, \hat{x}]}_{-i\hbar} + m^2\omega_0^2\hat{x}^2\right) \\ &= \hat{H} - \frac{1}{2}\hbar\omega_0.\end{aligned}$$

So, we have

$$\hat{H} = \hat{A}^\dagger\hat{A} + \frac{1}{2}\hbar\omega_0.$$

2 Finding energy eiegenstates

If we recall, for an eigenstate $|\psi\rangle$, we have $E = \langle\psi|\hat{H}|\psi\rangle$, with normalized $|\psi\rangle$, then we see for this case

$$E=\langle\psi|\hat{H}|\psi\rangle=\langle\psi|\hat{A}^\dagger\hat{A}|\psi\rangle+\frac{1}{2}\hbar\omega_0$$
$$\Rightarrow\ E\geq\frac{1}{2}\hbar\omega_0,\text{ since }\langle\psi|\hat{A}^\dagger\hat{A}|\psi\rangle=\|\hat{A}|\psi\rangle\|^2\geqslant 0$$

Hence, if we can find a state with $\hat{A}|0\rangle=0$, then this would be the ground state and we would have $E_{gs}=\frac{1}{2}\hbar\omega_0$.
For now, we assume such a state exists. Later, we will see that it does.

We next work out the intertwining relation (moving $\hat{A}^\dagger$ through $\hat{H}\Big)$. We start by writing out the product with the raising operator on the right, and then use the commutator to change the order of the pair of operators on the right. So, we have

$$\hat{H}\hat{A}^\dagger=\left(\hat{A}^\dagger\hat{A}+\frac{1}{2}\hbar\omega_0\right)\hat{A}^\dagger=\hat{A}^\dagger\left(\underbrace{\hat{A}^\dagger\hat{A}+\left[\hat{A},\hat{A}^\dagger\right]}_{\hat{A}\hat{A}^\dagger}+\frac{1}{2}\hbar\omega_0\right).$$

But, we have

$$\left[\hat{A},\hat{A}^\dagger\right]=\frac{1}{2m}\left[\hat{p}-im\omega_0\hat{x},\hat{p}+im\omega_0\hat{x}\right]=\frac{1}{2m}2im\omega_0[\hat{p},\hat{x}]=\hbar\omega_0,$$

so, we have

$$\hat{H}\hat{A}^\dagger=\hat{A}^\dagger\left(\hat{H}+\hbar w_0\right).$$

In words, moving the harmonic oscillator Hamiltonian past a raising operator originally on the right, shifts the Hamiltonian by $\hbar\omega_0$.

We use this to find all of the hiher-energy eigenstates. Our claim is that $\left(\hat{A}^\dagger\right)^n|0\rangle$ is an energy eigenstate. Our proof just uses the intertwining relation.
Proof: $\hat{H}(\hat{A}^\dagger)^n|0\rangle=\hat{H}\hat{A}^\dagger(\hat{A}^\dagger)^{n-1}=\hat{A}^\dagger\underbrace{(\hat{H}+\hbar\omega_0)(\hat{A}^\dagger)^{n-1}}_{\text{repeat n-1 more times}}|0\rangle$

$$=\left(\hat{A}^\dagger\right)^n\left(\hat{H}+n\hbar\omega_0\right)|0\rangle=\left(\hat{A}^\dagger\right)^n\left(\frac{1}{2}\hbar\omega_0+n\hbar\omega_0\right)|0\rangle$$
$$=\left(n+\frac{1}{2}\right)\hbar\omega_0\left(\hat{A}^\dagger\right)^n|0\rangle=E_n\left(\hat{A}^\dagger\right)^n|0\rangle.$$

So

$$E_n=\left(n+\frac{1}{2}\right)\hbar w_0.$$

We also use this to normalize the energy eigenstate. To do this, we identify an $\hat{A}\hat{A}^\dagger$ term in the center of the string of operators in the norm. We replace it by the Hamiltonian, and then use intertwining to evaluate it against the state on the right. We have

$$\begin{aligned}\langle 0|(\hat{A})^n(\hat{A}^\dagger)^n|0\rangle &=\langle 0|(\hat{A})^{n-1}\hat{A}\hat{A}^\dagger(\hat{A}^\dagger)^{n-1}|0\rangle\\ &=\langle 0|(\hat{A})^{n-1}\left(\hat{H}+\frac{1}{2}\hbar\omega_0\right)(\hat{A}^\dagger)^{n-1}|0\rangle\\ &=\langle 0|(\hat{A})^{n-1}(\hat{A}^\dagger)^{n-1}\left(\hat{H}+\left(n-\frac{1}{2}\right)\hbar\omega_0\right)|0\rangle\\ &=n\hbar\omega_0\langle 0|(\hat{A})^{n-1}\left(\hat{A}^\dagger\right)^{n-1}|0\rangle\end{aligned}$$

Repeat $n-1$ more times:

$$= n!\left(\hbar\omega_0\right)^n \underbrace{\langle 0 \mid 0\rangle}_{\text{assume normalized}} .$$

So

$$\boxed{|n\rangle = \frac{\left(\hat{A}^\dagger\right)^n}{\sqrt{n!}\left(\hbar\omega_0\right)^{n/2}}|0\rangle \text{ has } E_n = \hbar\omega_0\left(n+\frac{1}{2}\right).}$$

This may not look like what you are used to. It is conventional to redefine the operators via

$$\hat{a}^\dagger = \frac{-i}{\sqrt{\hbar\omega_0}}\hat{A}^\dagger \qquad \hat{a} = \frac{i}{\sqrt{\hbar\omega_0}}\hat{A}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{x} - i\frac{1}{m\omega_0}\hat{p}\right) \quad \hat{a} = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{x} + i\frac{1}{m\omega_0}\hat{p}\right).$$

Then, $\left[\hat{a},\hat{a}^\dagger\right] = 1, \hat{H} = \hbar\omega_0\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right), \hat{a}|0\rangle = 0, |n\rangle = \underbrace{\frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle}_{\text{we removed an inconsequential phase of } (-i)^n}$

3 Wavefunctions

Next up, we calculate the wave function. Recall, the eigenstate of position satisfies

$$\hat{x}|x\rangle = x|x\rangle,$$

which is an operator acting on a state giving us a number times that state.

Claim:

$$|x\rangle = e^{-\frac{i}{\hbar}\overbrace{x}^{\text{number}}\overbrace{\hat{p}}^{\text{operator}}}|x{=}0\rangle.$$

Proof:

$$\hat{x}|x\rangle = \underbrace{1}_{\text{insert } e^{-i\frac{x\hat{p}}{\hbar}}e^{i\frac{x\hat{p}}{\hbar}}}\hat{x}\left(e^{-i\frac{x\hat{p}}{\hbar}}|x{=}0\rangle\right)$$

$$\hat{X}|x\rangle = e^{ix\frac{\hat{p}}{\hbar}}\underbrace{e^{i\frac{x\hat{p}}{\hbar}}\hat{X}e^{-i\frac{x\hat{p}}{\hbar}}}_{\text{Hadamard}}|x{=}0\rangle$$

$$= e^{-ix\frac{\hat{p}}{\hbar}}\left(\hat{x} + \frac{ix}{\hbar}[\hat{p},\hat{x}] + \frac{1}{2}\left(\frac{ix}{\hbar}\right)^2[\hat{p},[\hat{p},\bar{x}]] + \cdots\right)|x{=}0\rangle$$

$$= e^{-i\frac{x\hat{p}}{\hbar}}(\hat{x}+x)|x{=}0\rangle.$$

But, $\hat{x}|x{=}0\rangle = 0\,|x{=}0\rangle{=}0 \quad \Rightarrow$

$$\hat{x}|x\rangle = xe^{-i\frac{x\hat{p}}{\hbar}}|x{=}0\rangle = x|x\rangle$$

So it is an eigenfunction!

The operator $e^{-i\frac{x\hat{p}}{\hbar}}$ is called the translation operator.

Before we calculate the wavefunction, it is worthwhile to talk about what it really is. The wavefunction is constructed by the overlap of two eigenfunctions from non-commuting operators. We should not think of this as a physical state the particle is in inbetween measurements. It is instead a calculational tool used to determine the results of experiments. Oftentimes, conventional QM instruction overemphasizes the importance of the wavefunction in coordinate space. You should not. We can interpret the overlap in two ways. For example $\psi_n(x) = \langle x|n\rangle$ can be thought of as the probability amplitude to find a particle that has energy E_n to be found in the region near x (we assume the energies are nondegenerate for simplicity here). Similarly, the probability amplitude (technically the complex conjugate of the probability amplitude) to find a particle located near x to have energy E_n. It is important to note that

$$|\langle x|n\rangle|^2 = |\langle n|x\rangle|^2,$$

so the probabilities of both statements are the same.

It is easy to overemphasize the importance of $\psi(x)$, even though we can also find $\psi(p)$ and other wavefunctions. When we look at this from an operator perspective, we will see that we can employ translation to relate the amplitude at the origin to the ampitude anywhere else. Amazingly, the results come entirely from operator algebra and the fact that $[\hat{x}, \hat{p}] = i\hbar$. In particular, we do not need the Schrodinger equation or any other differential equation to tell us how to determine the wavefunction. It follows from the properties of the ground state and the ladder operators. This holds not just for the simple harmonic oscillator, but we will see it holds for all solvable problems later in the course!

Now we calculate the wave function of the simple harmonic oscillator in coordinate space:

$$\psi_n(x) = \langle x|n\rangle = \langle x{=}0|e^{i\frac{x\hat{p}}{\hbar}} \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle.$$

But, $\hat{p} = \frac{\left(\hat{a}-\hat{a}^\dagger\right)m\omega_0}{2i}\sqrt{\frac{2\hbar}{m\omega_0}} = -i\sqrt{\frac{\hbar m\omega_0}{2}}\left(\hat{a}-\hat{a}^\dagger\right)$. So, the operator $e^{i\frac{x\hat{p}}{\hbar}} = e^{x\sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{a}-\hat{a}^\dagger\right)}$.

The question we now have is how do we use these operators to get the wavefunction? We only know two things about the states

$$\hat{a}|0\rangle = 0 \text{ and } \hat{x}|x{=}0\rangle = 0.$$

This means

$$e^{\alpha\hat{a}}|0\rangle = |0\rangle \text{ for any } \alpha$$

and

$$e^{\beta\hat{x}}|x{=}0\rangle = |x{=}0\rangle \text{ for any } \beta.$$

These relations ane very important. Always look for such annihilation relations in your work!

This then leads to a potential strategy to simplify the matrix element that gives the wavefunction. We need to convert $e^{\alpha\hat{p}}$ into some kind of $e^{\alpha'\hat{x}}$. Recalling that $\hat{p} \propto \hat{a} - \hat{a}^\dagger$ and $\hat{x} \propto \hat{a} + \hat{a}^\dagger$ tells us we should try the following:
1.) split up $e^{i\hat{p}} = e^{-\alpha\hat{a}^\dagger}e^{\alpha\hat{a}}\times$ correction terms
2.) move $e^{\alpha\hat{a}}$ to the right until it disappears when it hits $|0\rangle$

3.) replace it by $e^{-\alpha\hat{a}}|0\rangle = |0\rangle$
4.) move to the left until next to $e^{-\alpha a^\dagger}$
5.) bring into the same exponent $e^{-c(\hat{a}+\hat{a}^\dagger)}\times$ correction terms
6.) operate onto $\langle x{=}0|$ where $\langle x{=}0|e^{-c(\hat{a}+\hat{a}^\dagger)} = \langle x{=}0|$.

This will get rid of the full exponential form. Then we need to determine how to deal with the rest of the expression. To do that we work again with the same facts $\hat{a}|0\rangle = 0 \quad \langle x{=}0|\hat{x} = 0$ and use them to simplify until we get the final wavefunction.

Now, we go through the technical details carefully.

$$\text{Recall } e^{\frac{ix\hat{p}}{\hbar}} = \exp\left[x\sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{a} - a^\dagger\right)\right]$$
$$= \exp\left[x\sqrt{\frac{m\omega_0}{2\hbar}}\left(\underbrace{-\hat{a}^\dagger}_{\hat{A}} + \underbrace{\hat{a}}_{\hat{B}}\right)\right].$$

Recall as well $\left[\hat{a},\hat{a}^\dagger\right] = 1$, so use BCH

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]} \text{ or } e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]} = e^{\hat{A}+\hat{B}}.$$

Here, $\hat{A} = -x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}^\dagger \quad \hat{B} = x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a} \quad [\hat{A},\hat{B}] = \frac{mw_0}{2\hbar}x^2$.

So

$$\psi_n(x) = \langle x{=}0|\frac{e^{-\frac{m\omega_0 x^2}{4\hbar}}}{\sqrt{n!}}e^{-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}^\dagger}e^{x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}}(\hat{a}^\dagger)^n \underbrace{1}_{e^{-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}}e^{x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}}} |0\rangle$$

$$= \frac{e^{-\frac{m\omega_0 x^2}{4\hbar}}}{\sqrt{n!}}\langle x=0|e^{-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}^\dagger}\underbrace{\left(e^{x\sqrt{\frac{m\omega_0 x^2}{2\hbar}}\hat{a}}\hat{a}^\dagger e^{-x\sqrt{\frac{m\omega_0 x^2}{2\hbar}}\hat{a}}\right)^n}_{\text{Hadamard}}\underbrace{e^{x\sqrt{\frac{m\omega_0 x^2}{2\hbar}}\hat{a}}|0\rangle}_{|0\rangle}$$

$$= \frac{e^{-\frac{m\omega_0 x^2}{4\hbar}}}{\sqrt{n!}}\langle x{=}0|e^{-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}^\dagger}\underbrace{1}_{e^{-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}}e^{x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}}}\left(\hat{a} + x\sqrt{\frac{m\omega_0}{2\hbar}}\right)^n \underbrace{|0\rangle}_{\text{replace with } e^{-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}}|0\rangle}$$

$$= \frac{e^{-\frac{m\omega_0 x^2}{4\hbar}}}{\sqrt{n!}}\langle x=0|e^{-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}^\dagger}e^{-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}}\left(\underbrace{e^{x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}}\hat{a}^\dagger e^{-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}}}_{\text{Hadamard}} + x\sqrt{\frac{m\omega_0}{2\hbar}}\right)^n |0\rangle$$

$$= \frac{e^{-\frac{m\omega_0 x^2}{4\hbar}}}{\sqrt{n!}}\langle x=0|\underbrace{e^{-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}^\dagger}e^{-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}}}_{\text{BCH } \hat{A}=-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}^\dagger \quad \hat{B}=-x\sqrt{\frac{m\omega_0}{2\hbar}}\hat{a}}\left(\hat{a}^\dagger + \underbrace{x\sqrt{\frac{2m\omega_0}{\hbar}}}_{\text{twice as much}}\right)^n |0\rangle$$

$$= \frac{e^{-\frac{m\omega_0 x^2}{4\hbar}}}{\sqrt{n!}}\langle x=0|e^{-x\sqrt{\frac{m\omega_0}{2\hbar}}(\hat{a}+\hat{a}^\dagger)-\frac{1}{2}\frac{m\omega_0}{2\hbar}x^2}\left(\hat{a}^\dagger + x\sqrt{\frac{2m\omega_0}{\hbar}}\right)^n |0\rangle.$$

Recall: $\hat{a}^\dagger + \hat{a} = \sqrt{\frac{2m\omega_0}{\hbar}}\hat{x}$

$$= \frac{\overbrace{e^{-\frac{m\omega_0 x^2}{2\hbar}}}^{\text{twice as large an exponent}}}{\sqrt{n!}} \langle x{=}0| e^{-x\frac{m\omega_0}{\hbar}\hat{x}} \left(\hat{a}^\dagger + x\sqrt{\frac{2m\omega_0}{\hbar}}\right)^n |0\rangle.$$

But, $\hat{x}|x{=}0\rangle = 0 \Rightarrow \langle x{=}0|e^{-x\frac{m\omega_0}{\hbar}\hat{x}} = \langle x{=}0|$, so

$$\boxed{\psi_0(x) = \frac{e^{\frac{-m\omega_0 x^2}{2\hbar}}}{\sqrt{n!}} \langle x{=}0| \left(\hat{a}^\dagger + x\sqrt{\frac{2m\omega_0}{\hbar}}\right)^n |0\rangle}.$$

Lets look at $n = 0$ and $n = 1$ first:

$$n = 0 \quad \boxed{\psi_0(x) = e^{-\frac{1}{2}\frac{m\omega_0 x^2}{\hbar}} \langle x{=}0 \mid 0\rangle}$$

$$n = 1 \quad \psi_1(x) = e^{-\frac{1}{2}\frac{m\omega_0 x^2}{\hbar}} \langle x{=}0| \left(\hat{a}^\dagger + x\sqrt{\frac{2m\omega_0}{\hbar}}\right) |0\rangle$$

We will use "add zero" to convert the raising operator into an operator proportional to a position operator via $\hat{a}^\dagger = \hat{a}^\dagger + \hat{a} - \hat{a} = \sqrt{\frac{2m\omega_0}{\hbar}}\hat{x} - \hat{a}$. So, we have

$$\psi_1(x) = e^{-\frac{1}{2}\frac{m\omega_0}{x}x^2} \langle x{=}0| \left(\underbrace{\sqrt{2m\omega_0}\ \hat{x}}_{\text{gives 0 on left}} - \underbrace{\hat{a}}_{\text{gives 0 on right}} + x\sqrt{\frac{2m\omega_0}{\hbar}}\right) |0\rangle$$

$$\boxed{\psi_1(x) = e^{-\frac{1}{2}\frac{m\omega_0 x^2}{\hbar}} x\sqrt{\frac{2mw_0}{\hbar}} \langle x{=}0|0\rangle.}$$

Define $H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}x\right) = \frac{\sqrt{2^n}}{\underbrace{\langle x{=}0|0\rangle}_{\text{factors need to relate to Hermite's work}}} \langle x{=}0| \left(\hat{a}^\dagger + x\sqrt{\frac{2m\omega_0}{\hbar}}\right)^n |0\rangle$. We just showed that $H_0 = 1$ and $H_1 = 2\sqrt{\frac{m\omega_0}{\hbar}}x$ from our calculations of ψ_0 and ψ_1. We now find a recurrence relation for general n. We first split off one factor in the matrix element to the left, and rework it as we did for the first excited state:

$$H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}x\right) = \frac{\sqrt{2^n}}{\langle x{=}0|0\rangle} \langle x{=}0| \left(\hat{a}^\dagger + x\sqrt{\frac{2m\omega_0}{\hbar}}\right) \left(\hat{a}^\dagger + x\sqrt{\frac{2m\omega_0}{\hbar}}\right)^{n-1} |0\rangle$$

$$= \frac{\sqrt{2^n}}{\langle x{=}0|0\rangle} \langle x{=}0|\hat{a}^\dagger \left(\hat{a}^\dagger + x\sqrt{\frac{2m\omega_0}{\hbar}}\right)^{n-1} |0\rangle + 2x\sqrt{\frac{m\omega_0}{\hbar}} H_{n-1}\left(\sqrt{\frac{m\omega_0}{\hbar}}x\right).$$

But, $\langle x{=}0|\hat{a}^\dagger = \langle x{=}0|(-\hat{a})$ since $\langle x{=}0|\underbrace{(\hat{a}+\hat{a}^\dagger)}_{\text{prop. to }\hat{x}} = 0$ and

$$\hat{a}\left(\hat{a}^\dagger + x\sqrt{\frac{2m\omega_0}{\hbar}}\right)^{n-1}|0\rangle = \left[\hat{a}, \left(\hat{a}^\dagger + x\sqrt{\frac{2m\omega_0}{\hbar}}\right)^{n-1}\right]|0\rangle \quad \text{since } \hat{a}|0\rangle = 0$$

$$= (n-1)\left(\hat{a}^\dagger + x\sqrt{\frac{2m\omega_0}{\hbar}}\right)^{n-2}|0\rangle$$

So $\boxed{H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}x\right) = 2\sqrt{\frac{m\omega_0}{\hbar}}xH_{n-1}\left(\sqrt{\frac{m\omega_0}{\hbar}}x\right) - 2(n-1)H_{n-2}\left(\sqrt{\frac{m\omega_0}{\hbar}}x\right)}$

This, combined with the H_0 and H_1 values already found are the recurrence relations for the Hermite polynominals, summarized in the following table.

n	$H_n(y)$
0	1
1	$2y$
2	$4y^2 - 2$
3	$8y^3 - 12y$
4	$16y^4 - 48y^2 + 12$
5	$32y^5 - 160y^8 + 120y$

Note, this is the physicist's convention. Mathematicians use a different one.

The last thing we do is normalize the wavefunction. This requires us to normalize just the ground state, because we have otherwise been working with normalized states.

$$\int_{-\infty}^{+\infty} \psi_0^2(x) = 1 = \int_{-\infty}^{+\infty} e^{-\frac{m\omega_0}{\hbar}x^2}|\langle x{=}0|0\rangle|^2$$

$$= \sqrt{\frac{\pi\hbar}{m\omega_0}}|\langle x{=}0|0\rangle|^2$$

$$\Rightarrow \boxed{\langle x{=}0|0\rangle = \left(\frac{m\omega_0}{\hbar\pi}\right)^{\frac{1}{4}}}$$

4 Uncertainty

Everything is done algebraically and comes from operators! No series solutions of differential equations!

We end by examining uncertainty $\quad \hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a}+\hat{a}^\dagger)$

$$(\Delta x)_n^2 = \langle n|\hat{x}^2|n\rangle - (\langle n|\hat{x}|n\rangle)^2$$
$$= \frac{\hbar}{2m\omega_0}\langle n|\left(\hat{a}+\hat{a}^\dagger\right)^2|n\rangle - \frac{\hbar}{2m\omega_0}\left(\langle n|\left(\hat{a}+\hat{a}^\dagger\right)|n\rangle\right)^2$$
$$= \frac{\hbar}{2m\omega_0}\frac{1}{n!}\langle 0|(\hat{a})^n\left(\hat{a}+\hat{a}^\dagger\right)^2\left(\hat{a}^\dagger\right)^n|0\rangle$$
$$- \frac{\hbar}{2m\omega_0}\frac{1}{(n!)^2}\left(\langle 0|(\hat{a})^n \underbrace{\left(\hat{a}+\hat{a}^\dagger\right)}_{=0 \text{ since we cannot pair all } \hat{a} \text{ and } \hat{a}^\dagger} \left(\hat{a}^\dagger\right)^n|0\rangle\right)^2$$
$$= \frac{\hbar}{2m\omega_0 n!}\langle 0|(\hat{a}^n(\underbrace{\hat{a}^2}_{0} + \hat{a}\hat{a}^\dagger + \underbrace{\hat{a}^\dagger\hat{a}}_{=\hat{a}\hat{a}^\dagger-1} + \underbrace{(\hat{a}^\dagger)^2}_{0})(\hat{a}^\dagger)^n|0\rangle$$
$$\boxed{(\Delta x)_n^2 = \frac{\hbar}{2m\omega_0}(2n+1)}$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega_0}{2}}(\hat{a}-\hat{a}^\dagger)$$
$$(\Delta p)_n^2 = -\frac{\hbar m\omega_0}{2}[\langle n|(\hat{a}-\hat{a}^\dagger)^2|n\rangle - \underbrace{(\langle n|(\hat{a}-\hat{a}^\dagger)|n\rangle)^2}_{0}]$$
$$\text{and we have } \langle n|\underbrace{\hat{a}^2}_{0} - 2\hat{a}\hat{a}^\dagger + \underbrace{(\hat{a}^\dagger)^2}_{0}|n\rangle = -(2n+1)$$
$$\boxed{(\Delta p)_n^2 = \frac{\hbar m\omega_0}{2}(2n+1)}$$

So we find that $\boxed{(\Delta x)_n(\Delta p)_n = \frac{\hbar}{2}(2n+1)}$.

The uncertainty is minimal for the ground state but grows with n.