Phys 506 lecture 34: What is a photon?

1 Classical Maxwell equations

In the next two lectures, we will summarize quantum optics with the goals of (1) establishing precisely what a photon is and (2) describing how quantum optics principles are employed in the LIGO experiment to improve the precision of the measurements. It will be a crash course. You need to review ladder operators for the simple harmonic oscillator and coherent states for this lecture.

We begin with classical description of an electric field given by

$$\mathbf{E}(r,t) = \sum_l \varepsilon_l E_l(t) e^{i\mathbf{k}_l \cdot \mathbf{r}} + \text{c.c.}$$

where we are describing a real traveling wave. Here l denotes the mode described by the wavevector $\mathbf{k}_l$ and polarization ε_l. Here, we will focus on linear polarization only. The function $E_l(t)$ can be complex-valued.

We require the wave to satisfy Maxwell's equations.

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = 0, \quad \nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \quad \nabla \cdot \mathbf{B}(\mathbf{r},t) = 0, \quad \nabla \times \mathbf{B}(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}$$

Using the fact that $\nabla e^{i\mathbf{k}_l \cdot \mathbf{r}} = i\mathbf{k}_l e^{i\mathbf{k}_l \cdot \mathbf{r}}$ and $\nabla \cdot \mathbf{E} = 0$, we can say that $\mathbf{k}_l \perp \varepsilon_l$. Also, $\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \implies \mathbf{B}(\mathbf{r},t) \propto \mathbf{k}_l \times \varepsilon_l$. Then $\nabla \cdot \mathbf{B} = 0$ is automatically satisfied. We end up with

$$\frac{\partial^2 E_l(t)}{\partial t^2} = -\frac{k_l^2}{c^2} E_l(t).$$

Define the angular frequency $\omega_l = \frac{k_l}{c}$. Then $E_l(t) = E_l(0) e^{-i\omega_l t}$.

$$\implies \mathbf{B}(\mathbf{r},t) = \sum_l \frac{\mathbf{k}_l \times \varepsilon_l}{\omega_l} E_l(t) e^{i\mathbf{k}_l \cdot \mathbf{r}} + \text{c.c.}$$

2 Quantizing the fields

We define the single photon amplitude as $\varepsilon_l^{(1)} = \sqrt{\frac{\hbar\omega_l}{2\epsilon_0 V_l}}$ and introduce V_l as the quantization volume for mode l. We will be working in a volume L^3 with periodic boundary conditions so $\mathbf{k}_l = \frac{2\pi}{L}(n_x, n_y, n_z)$ for $n_\alpha \in \mathbb{Z}$. The polarization is one of the two directions perpendicular to k_l. Note that we use $\mathbf{k}_l, \varepsilon_l$, and $\mathbf{k}_l \times \varepsilon_l$ as a triad to define the coordinates of the three dimensions.

We also choose $\varepsilon_{-l} = \varepsilon_l$ when $l : \frac{2\pi}{L}(n_x, n_y, n_z; \varepsilon_l)$ and $-l : \frac{2\pi}{L}(-n_x, -n_y, -n_z; \varepsilon_{-l})$. This will help us with calculating the total energy. But before that, we note two more things. We will be writing $E_l(t) = \varepsilon_l \alpha_l(t)$ with $\alpha_l(t)$ possibly complex. The we also define the two *quadrature parameters*

$$Q_l(t) = \sqrt{\frac{\hbar}{2}}(\alpha_l(t) + \alpha_l^*(t))$$
$$P_l(t) = -i\sqrt{\frac{\hbar}{2}}(\alpha_l(t) - \alpha_l^*(t))$$

The energy is given by

$$E = \frac{\varepsilon_0}{2}\int d^3r\ (\mathbf{E}^2 + c^2\mathbf{B}^2)$$

In calculating this, we note that

$$\int d^3r\ e^{i\mathbf{k}_l\cdot\mathbf{r} - \mathbf{k}_{l'}\cdot\mathbf{r}} = \delta_{n_l,n_{l'}} V.$$

We end up with two types of terms: Those with $l = l'$ and those with $l = -l'$. Then (you should do this),

$$\begin{aligned}
E &= \frac{\varepsilon}{2} V \sum_l \left(2|E_l(t)|^2 - E_l(t)E_{-l}(t) - E_l^*(t)E_{-l}^*(t) + 2|E_l(0)|^2 + E_l(t)E_{-l}(t) + E_l^*(t)E_{-l}^*(t)\right)\\
&= 2\varepsilon_0 V \sum_l (\varepsilon_l)^2 |\alpha_l|^2\\
&= \sum_l \hbar\omega_l |\alpha_l|^2
\end{aligned}$$

Now we quantize: Let $Q_l \to \hat{Q}_l$ and $P_l \to \hat{P}_l$ with $[\hat{Q}_l, \hat{P}_{l'}] = i\hbar\delta_{ll'}$. Then define

$$\hat{a}_l = \frac{1}{\sqrt{2\hbar}}(\hat{Q}_l + i\hat{P}_l)$$
$$\hat{a}_l^\dagger = \frac{1}{\sqrt{2\hbar}}(\hat{Q}_l - i\hat{P}_l).$$

Written in terms of the quadratures,

$$H = \sum_l \frac{\omega_l}{2}(\hat{Q}_l^2 + \hat{P}_l^2),$$

which shows the relationship with the simple harmonic oscillator. Back to the field, we now have an operator

$$\hat{\mathbf{E}}(\mathbf{r}) = \hat{E}^{(+)} + \hat{E}^{(-)} = i\sum_l \varepsilon_l \hat{a}_l e^{i\mathbf{k}_l\cdot\mathbf{r}} + \text{h.c.}$$

The eigenstates are labeled by number operator eigenstates

$$|n_1, n_2, \ldots\rangle = \frac{(\hat{a}_1^\dagger)^{n_1}}{\sqrt{n_1!}}\frac{(\hat{a}_2^\dagger)^{n_2}}{\sqrt{n_2!}}\ldots|0\rangle$$

and $E_{n_1\cdots} = \sum_l \hbar\omega_l\left(n_l + \frac{1}{2}\right)$. Usually, we only include nonzero n_l's in the labeling. Note the vacuum has "infinite energy" given by $\sum_l \hbar\omega_l \frac{1}{2}$. But we avoid this energy by focusing on the excitation energy with respect to the vacuum given by $E^{ex} = \sum_l \hbar\omega_l n_l$.

We will be working in the Heisenberg representation where $\hat{a}_l(t) = \hat{a}_l e^{-i\omega_l t}$ and $\hat{a}_l^\dagger(t) = \hat{a}_l^\dagger e^{i\omega_l t}$. Now consider a single mode state $|n_l\rangle$. The average of the electric field vanishes

$$\left\langle n_l \middle| \hat{\mathbf{E}}(\mathbf{r},t) \middle| n_l \right\rangle = \left\langle n_l \middle| \hat{a}_l e^{ik\cdot r - i\omega_l t} - \hat{a}_l^\dagger e^{-ik\cdot r + i\omega_l t} \middle| n_l \right\rangle = 0$$

since $\hat{a}$ and $\hat{a}^\dagger$ operators are unbalanced. We calculate the fluctuations from

$$\left\langle n_l \middle| \hat{\mathbf{E}} \cdot \hat{\mathbf{E}} \middle| n_l \right\rangle = -\varepsilon_l \cdot \varepsilon_l (\varepsilon_l^{(1)})^2 \left\langle n_l \middle| \hat{a}_l^2 e^{i(2k_l\cdot r - 2\omega_l t)} - \hat{a}_l \hat{a}_l^\dagger - \hat{a}_l^\dagger \hat{a}_l + \hat{a}_l^{\dagger 2} e^{-i(2k_l\cdot r - 2\omega_l t)} \middle| n_l \right\rangle$$

$$= (\varepsilon_l^{(1)})^2 (2n_l + 1)$$

so $\Delta\mathbf{E} = \varepsilon_l^{(1)}\sqrt{2n_l+1}$. In particular, the vacuum ($n_l = 0$) has fluctuations. This is real and responsible for spontaneous emission, Lamb shift, g-2, Casimir effect, etc. A similar calculation shows that

$$\left\langle 0 \middle| \hat{P}_l \middle| 0 \right\rangle = \left\langle 0 \middle| \hat{Q}_l \middle| 0 \right\rangle = 0$$

$$\left\langle 0 \middle| \hat{P}_l^2 \middle| 0 \right\rangle = \left\langle 0 \middle| \hat{Q}_l^2 \middle| 0 \right\rangle = \sqrt{\frac{\hbar}{2}}$$

so $\Delta Q_l \Delta P_l = \frac{\hbar}{2}$, which is a minimum uncertainty state (same as the ground state).

3 Detecting photons

Our next step is to describe photodetection. We will describe a photomultiplier tube which uses the photoelectric effect and a cascade.

A single photon releases an electron which is accelerated and leads to huge amplification of electrons which can be measured as an electron pulse. The important aspects are that these detectors are efficient and fast (there are more efficient fast detectors, but we focus on these which is all we need).

Suppose light is traveling with a areal profile of S and impinging on detectors 1 and 2. The probability to detect one photon is $dP(\mathbf{r},t) = W^{(1)}(r,t)dS\ dt$ with $W^{(1)}(r,t) = s|\hat{E}^+(\mathbf{r},t)|\psi\rangle|^2$, which is the single-photon detection probability. s is the sensitivity of the detector and $|\psi\rangle$ is the photon state.

The probability to detect two photons is $dP(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2) = W^{(2)}(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2)dS\ dt$ with

$$W^{(2)}(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2) = S^2|\hat{E}^+(\mathbf{r}_2,t_2)\hat{E}^+(\mathbf{r}_1,t_1)|\psi\rangle|^2$$

and

$$\hat{E}_l^+(\mathbf{r},t) = i\varepsilon_l\varepsilon_l^{(1)}\hat{a}_l e^{i(\mathbf{k}_l\cdot\mathbf{r} - i\omega_l t)}$$

Consider a single photon state. We have $|1_l\rangle = \hat{a}_l^\dagger|0\rangle$. Then,

$$W^{(1)}(\mathbf{r},t) = s|i\varepsilon_l\varepsilon_l^{(1)}e^{i(\mathbf{k}_l\cdot\mathbf{r} - i\omega_l t)}\hat{a}_l\hat{a}_l^\dagger|0\rangle|^2$$

But

$$\hat{a}_l \hat{a}_l^\dagger |0\rangle = [\hat{a}_l, \hat{a}_l^\dagger] |0\rangle = |0\rangle$$

so $W^{(1)}(\mathbf{r}, t) = s$. Similarly, we have that

$$W^{(2)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) = s^2 |i\varepsilon_l \varepsilon_l^{(1)} e^{i(\mathbf{k}_l \cdot \mathbf{r} - i\omega_l t)} i\varepsilon_l \varepsilon_l^{(1)} e^{i(\mathbf{k}_l \cdot \mathbf{r} - i\omega_l t)} \hat{a}_l \hat{a}_l \hat{a}_l^\dagger |0\rangle |^2 = 0,$$

since $\hat{a}_l \hat{a}_l \hat{a}_l^\dagger |0\rangle = 0$. So a single photon can only be measured once! Measuring one photon alters the quantum state so it cannot be measured again. This is, in many respects, one of the critical aspects of what a photon is.

4 Making a single photon source

We now describe how one makes a single photon source. Here is the two photon excitation process, and a two photon decay. The two photons emerge within a few nanoseconds. So, by observing ω_H, one is sure a photon ω_0 is emitted within the next 10-15 ns.

Collect the atoms at the focus of a parabolic mirror. Then, the photon will "live" in a volume given by $S \subset T$ with T being the lifetime of the excited state. Recall

$$\frac{dP^{(1)}}{dt\, dS} = s|\varepsilon_l^{(1)}|^2 = \frac{s\hbar\omega_l}{2\varepsilon_0 V_l} = \frac{s\hbar\omega_l}{2\varepsilon_0 ScT}$$

So

$$\frac{dP^{(1)}}{dt} = \int dS\, \frac{dP^{(1)}}{dt\ S} = \frac{s\hbar\omega_l}{2\varepsilon_0 cT_l}$$

and

$$\int_T \frac{dP^{(1)}}{dt} = \frac{s\hbar\omega_l}{2\varepsilon c} = 1$$

for a perfect detector. So $S_{\text{perfect}} = \text{perfect efficiency} = \frac{2\varepsilon_0}{\hbar\omega_l}$. We define the quantum efficiency as

$$s = \eta S_{\text{perfect}}$$

Then, $\boxed{\omega^{(1)} = \frac{\eta}{ST}}$. The real photon emitted is in a wavepacket $|\psi\rangle = \sum_l c_l |1_l\rangle$ with

$$c_l = \frac{\kappa e^{i\omega t_0}}{\omega_l - \omega_0 + i\frac{\Gamma}{2}} = \text{Lorentzian lineshape}$$

κ = normalization constant, ω_0 = frequency of the excited state, Γ = lifetime = $\sqrt{\frac{c\pi}{L}}$.
Assume the photon is emitted at time t_0. It must travel a distance z to reach the detector. So,

$$W^{(1)}(z, t) = s|\hat{E}^+(z, t) |\psi(t_0)\rangle |^2 = s| \sum_l \varepsilon_l \varepsilon_l^{(1)} \hat{a}_l e^{-i\omega_l(-\frac{z}{2} + t)} \sum_{l'} \frac{\sqrt{\frac{c\Gamma}{L}} e^{i\omega_{l'} t_0}}{(\omega_{l'} - \omega_0) + i\frac{\Gamma}{2}} \hat{a}_{l'}^\dagger |0\rangle |^2$$

But $\hat{a}_l \hat{a}_{l'}^\dagger |0\rangle = \delta_{ll'} |0\rangle$ so

$$= s| \sum_l i\varepsilon_0 \varepsilon_l^{(1)} \sqrt{\frac{c\Gamma}{L}} \frac{e^{-i\omega_l(t - t_0 - \frac{z}{c})}}{(\omega_l - \omega_0) + i\frac{\Gamma}{2}} |0\rangle |^2.$$

Assuming $\varepsilon_l^{(1)}$ is small, since Γ is small, and transform the sum to an integral.

$$i\int_{-\infty}^{\infty} d\omega_l \frac{L}{2\pi c}\left(\frac{c\Gamma}{L}\right)^{1/2} \frac{e^{-i\omega_l(t-t_0-\frac{z}{c})}}{\omega_l-\omega_0+i\frac{\Gamma}{2}}|0\rangle = i\frac{L}{2\pi c}\left(\frac{c\Gamma}{L}\right)^{1/2} e^{-\omega_0(t-t_0-\frac{z}{c})}(-2\pi i\Theta(t-t_0-\frac{z}{c}))e^{-\frac{\Gamma}{2}(t-t_0-\frac{z}{c})}|0\rangle$$

and

$$W^{(1)}(z,t) = s\frac{\hbar\omega_0}{2\varepsilon_0 V}\frac{L\Gamma}{c}e^{-\Gamma(t-t_0-\frac{z}{c})}\Theta\left(t-t_0-\frac{z}{c}\right)$$

Recall $V = sL$, $s = \frac{2\varepsilon_0 S}{\hbar\omega_0}$. So,

$$W^{(1)}(z,t) = \eta\frac{\Gamma}{c}e^{-\Gamma(t-t_0-\frac{z}{c})}\Theta\left(t-t_0-\frac{z}{c}\right).$$

The probability exponentially decays away from the initial time which has been measured experimentally! These single photon sources can be verified by measuring them on a beam splitter. We analyze this next:

The beam splitter is a partially silvered mirror that reflects the amplitude with strength r (or $-r$) depending on which side (silvered or not) and with strength t. We have

$$\hat{E}_3 = r\hat{E}_1 + t\hat{E}_2 \text{ and } \hat{E}_4 = t\hat{E}_1 - r\hat{E}_2.$$

The probability in 3 and 4 for single-photon detection is $s|E_3+(r_3,t_3)|\psi_{in}\rangle|^2$ and $s|E_4^+(r_4,t_4)|\psi_{in}\rangle|^2$ where

$$|\psi_{in}\rangle = (\gamma|1\rangle_1 + \sqrt{1-\gamma^2}|0\rangle_1)\otimes|0\rangle_2$$

where the efficiency to detect the heralded photon is γ (not all photons are detected). One finds

$$\frac{dP_3(t)}{dt} = \eta_3|\gamma|^2|r|^2\Theta\left(t_3-t_0-\frac{z_3}{c}\right)e^{-\Gamma(t_3-t_0-\frac{z_3}{c})}$$
$$\frac{dP_4(t)}{dt} = \eta_4|\gamma|^2|r|^2\Theta\left(t_4-t_0-\frac{z_4}{c}\right)e^{-\Gamma(t_4-t_0-\frac{z_4}{c})}.$$

Integrating over a few $1/\Gamma$'s yields

$$N_3 = \eta_3|\gamma|^2|r|^2 \text{ and } N_4 = \eta_4|\gamma|^2|t|^2.$$

Coincidences are of course zero, but in a real experiment, we see coincidence due to dark background current and more than one atom spontaneously emitting in the measurement window. We define:

$$P_3 = \frac{N_3}{N_H},\ P_4 = \frac{N_4}{N_H},\text{ and } P_c\frac{N_c}{N_H}$$

where N_H is the number of heralded photons and

$$\alpha = \frac{P_c}{P_3P_4} = \frac{N_cN_H}{N_3N_4}.$$

For a quantum system, we have $\alpha < 1$. A classical system must have $\alpha > 1$, since

$$P_C = \langle W^{(2)}\rangle \geq \langle W^{(1)}\rangle^2 = P_3P_4.$$

We can characterize the single-photon quantum nature by observing $\alpha < 1$.

In summary, a one photon state, given by $|1\rangle = \sum_l c_l|1_l\rangle$ is an eigenstate of $\hat{N}$, but not necessarily $\hat{H}$ (it has definite particle number but not necessarily definite energy). When we measure it, it can be observed only once. It has an extent in time given by some small multiple of $1/\Gamma$ with high probability. Sources of single photons are not just very dim light, as we see next.

5 Semiclassical sources of light

We end the chapter with a discussion of semiclassical states of light, described by our old friend the coherent state, which satisfies

$$\hat{a}_l \left|\alpha_l\right\rangle = \alpha_l \left|\alpha_l\right\rangle$$

where

$$\left|\alpha_l\right\rangle = D(\alpha_l) \left|0\right\rangle = e^{\alpha_l \hat{a}_l + \alpha^* \hat{a}} \left|0\right\rangle .$$

For a semiclassical state, we have

$$\begin{aligned} W^{(1)}(\mathbf{r}, t) &= s|\hat{E}^{+}(\mathbf{r}, t) \left|\alpha_l\right\rangle|^2 \\ &= s(\varepsilon_l^{(l)})^2 |\alpha_l|^2 \\ W^{(2)}(\mathbf{r}, t, \mathbf{r}', t') &= s^2 |\hat{E}^{\dagger}(\mathbf{r}', t') \hat{E}^{+}(\mathbf{r}, t) \left|\alpha_l\right\rangle|^2 \\ &= s^2 (\varepsilon_l^{(2)})^4 |\alpha_l|^4 = W^{(1)}(\mathbf{r}, t) W^{(1)}(\mathbf{r}', t') \end{aligned}$$

This is the classical result for a classical field as well.

Note that it shows that even for very dim light, with much less than one photon in each measurement interval, we will sometimes observe two photons in one interval. Furthermore the α for semiclassical systems is 1. The experiment has been done and verified. This clarifies an old result where it was believed that classical light becomes a single photon source when very dim. But it never does.

Incandescent light, LEDs, lasers, are all sources of semiclassical light. Essentially all light sources we commonly use are semiclassical. Single photon light sources are much more difficult to make. Next time, we discuss squeezed light, measuring quadratures and how they improve LIGO.