Phys 506 lecture 4: Coherent States

Coherent states play two important roles. On the one hand they can be used to illustrate how close a quantum system can be made to look like the back and forth motion of a mass on a spring. On the other hand, they play a critical role in the quantization of light, with coherent states being the quantum representation of the electromagnetic fields in a laser. We won't be able to go into detail about that here, but we will develop these states for the SHO. They are our connection to the classical world. As a side note, I am currently working on generalizing this idea into other systems than just the simple harmonic oscillator, and it looks very promising.

1 Definition of a coherent state

There are many ways to motivate the coherent states, but here we de so as the eigenstates of the lowering operator $\hat{a} : |\alpha\rangle$ satisfies $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. But wait, you say, $\hat{a}$ is not Hermitian, how can it have eigenstates? It turns out it can, but some of our familiar properties do not hold. First, α can be complex and need not be real and second, the eigenstates are not orthogonal. They also are over complete, which we will describe below.

But first, let's refresh our memories about $\hat{a}$.

$$[\hat{H}, \hat{a}] = \hbar\omega_0 \left[\hat{a}^\dagger \hat{a} + \frac{1}{2}, \hat{a}\right] = -\hbar\omega_0 \hat{a} \quad \left(\text{since } \left[\hat{a}, \hat{a}^\dagger\right] = 1\right)$$
$$\left[\hat{H}, \hat{a}^\dagger\right] = \hbar\omega_0 \left[\hat{a}^\dagger \hat{a} + \frac{1}{2}, \hat{a}^\dagger\right] = \hbar\omega_0 \hat{a}^\dagger$$

These lead us to conclude that $\hat{a}$ lowers the energy of an energy eigenstate by $\hbar\omega_0$, while $\hat{a}^\dagger$raises it by $\hbar w_0$.

We also know $\left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] = n\left(\hat{a}^\dagger\right)^{n-1}$ and $\left[\hat{a}^\dagger, \left(\hat{a}\right)^n\right] = -n\hat{a}^{n-1}$ (you can use induction to show this). So

$$\hat{a}|n\rangle = \hat{a}\frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle = \frac{\left(\hat{a}\left(\hat{a}^\dagger\right)^n - \left(\hat{a}^\dagger\right)^n \overbrace{\hat{a}}^{\text{adding zero}}\right)}{\sqrt{n!}}|0\rangle \tag{1}$$
$$= \frac{\left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right]}{\sqrt{n!}}|0\rangle = \frac{n}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^{n-1}|0\rangle = \sqrt{n}|n-1\rangle. \tag{2}$$

Similarly $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ (but this proof is much easier since no commutator is needed, try it.)

Let's find $|\alpha\rangle$. Let $|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$ (possible since$\{\ |n\rangle\}$ is complete). Then

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} c_n \hat{a}|n\rangle = \sum_{n=1}^{\infty} c_n \underbrace{\sqrt{n}}_{\text{vanishes when n=0}} |n-1\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle$$
$$= \alpha|\alpha\rangle \Rightarrow c_{n+1}\sqrt{n+1} = \alpha c_n$$

$$\Rightarrow \quad c_0, \quad c_1 = \frac{\alpha}{\sqrt{1}}c_0, \quad c_2 = \frac{\alpha}{\sqrt{2}}c_1 = \frac{\alpha^2}{\sqrt{2\cdot 1}}c_0, \quad c_3 = \frac{\alpha^3}{\sqrt{3\cdot 2\cdot 1}}c_0,$$

and $c_n = \frac{\alpha^n}{\sqrt{n!}}c_0$. The coefficient c_0 is determined by normalization. Let's first compute the overlap $\langle\beta|\alpha\rangle$

$$\begin{aligned}\langle\beta|\alpha\rangle &= \sum_{n=0}^{\infty} \frac{(\beta^*)^n}{\sqrt{n!}} c_0^*(\beta)\langle n| \sum_{m=0}^{\infty} \frac{(\alpha)^m}{\sqrt{m!}} c_0(\alpha)|m\rangle \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta^*)^n (\alpha)^m}{\sqrt{n!m!}} c_0^*(\beta)c_0(\alpha) \underbrace{\langle n|m\rangle}_{\delta_{nm}} \\ &= \sum_{m=0}^{\infty} \frac{(\beta^*\alpha)^m}{m!} c_0^*(\beta)c_0(\alpha) = e^{\beta^*\alpha} c_0^*(\beta)c_0(\alpha).\end{aligned}$$

Set $\alpha = \beta \Rightarrow \langle\alpha|\alpha\rangle = e^{|\alpha|^2}|c_0\,(\alpha)|^2 = 1$

$$\Rightarrow |c_0(\alpha)| = e^{-\frac{1}{2}|\alpha|^2} \qquad \text{(pick } c_0 = \text{ real and positive)}$$

$$\boxed{|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{1}{2}|\alpha|^2}|n\rangle \quad \langle\alpha|\beta\rangle = e^{\beta^*\alpha - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2}}$$

Note that the overlap is non zero, but as $\alpha - \beta$ becomes large in magnitude, $\langle\alpha|\beta\rangle$ gets small very quickly. Hence coherent states are not orthogonal!

You should think about why $\hat{a}^\dagger$ has no eigenstate. Ask, if you de not see why it cannot work.

2 Displacement operator

Let us play with the representation of a coherent state

$$\begin{aligned}|\alpha\rangle &= \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{1}{2}|\alpha|^2}|n\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{1}{2}|\alpha|^2} \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\left(\alpha\hat{a}^\dagger\right)^n}{n!}|0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger}|0\rangle\end{aligned}$$

but recall $e^{-\alpha\hat{a}}|0\rangle = |0\rangle$ and recall $(\alpha\hat{a}^{+})^{\dagger} = \alpha^{*}\hat{a})$, so that

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^{\dagger}} e^{-\alpha^{*}\hat{a}}|0\rangle$$

Now use BCH with $\hat{A} = \alpha\hat{a}^{\dagger}$ and $\hat{B} = -\alpha^{*}\hat{a}$ and $[\hat{A}, \hat{B}] = |\alpha|^2$,

$$|\alpha\rangle = e^{-\frac{1}{2}\left|\alpha\right|^2} e^{\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a} + \frac{1}{2}|\alpha|^2}|0\rangle$$

$$\boxed{|\alpha\rangle = e^{\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}}|0\rangle}.$$

We define the displacement operator $\hat{D}(\alpha)$ to be $\hat{D}(\alpha) = e^{\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}}$

$$\boxed{|\alpha\rangle = \hat{D}(\alpha)|0\rangle}.$$

But recall that

$$\hat{a} = \sqrt{\frac{mw_0}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega_0}\hat{p}\right) \quad \hat{a}^{\dagger} = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega_0}\hat{p}\right)$$

So that $\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a} = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\alpha\hat{x} - \frac{i\alpha}{m\omega_0}\hat{p} - \alpha^{*}\hat{x} - \frac{i\alpha^{*}}{m\omega_0}\hat{p}\right)$

$$\boxed{\alpha\hat{a}^{\dagger} - \alpha^{\psi}\hat{a} = i\sqrt{\frac{2m\omega_0}{\hbar}}\,\mathrm{Im}(\alpha)\hat{x} - i\sqrt{\frac{2}{\hbar m\omega_0}}\,\mathrm{Re}(\alpha)\,\hat{p}}$$

So, if α is real $\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a} = -i\frac{x_0\hat{p}}{\hbar}$ $\quad x_0 = \sqrt{\frac{2\hbar}{m\omega_0}}\,\mathrm{Re}(\alpha)$ and
if α is imaginary $\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a} = i\frac{p_0\hat{x}}{\hbar}$ $\quad p_0 = \sqrt{2\hbar\pi\omega_0}\,\mathrm{Im}(\alpha)$

Hence, the displacement operator gives both a translation in space and a translation in momentum to the ground-state energy eigenfunction.

3 Working with coherent states

Calculating with coherent states is easy because

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \text{ and } \langle\alpha|\hat{a}^{\dagger} = \langle\alpha|\alpha^{*}.$$

But note that we do not know what $\hat{a}^{\dagger}|\alpha\rangle$ is, nor do we know $\langle\alpha|\hat{a}$. So

$$\langle\alpha|\hat{x}|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\langle\alpha|\hat{a} + \hat{a}^{\dagger}|\alpha\rangle = \sqrt{\frac{2\hbar}{m\omega_0}}\,\mathrm{Re}\,\alpha$$

Then for a more complicated calculation, we have

$$\begin{aligned}
\langle\alpha|\hat{x}^2|\alpha\rangle &= \frac{\hbar}{2m\omega_0}\langle\alpha|\left(\hat{a}^2 + \hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a} + (\hat{a}^{\dagger})^2\right)|\alpha\rangle \\
&= \frac{\hbar}{2m\omega_0}\langle\alpha|\left(\hat{a}^2 + \hat{a}^{\dagger}\hat{a} + \left[\hat{a}, \hat{a}^{\dagger}\right] + \hat{a}^{\dagger}\hat{a} + (\hat{a}^{\dagger})^2\right)|\alpha\rangle \\
&= \frac{\hbar}{2m\omega_0}\left(\alpha^2 + 2|\alpha|^2 + 1 + \alpha^{*2}\right)
\end{aligned}$$

Note how we had to "normal-order" the operators with daggers to the left, in order to evaluate the matrix element. This then gives for uncertainty

$$\begin{aligned}(\Delta x)^2_\alpha &= \langle\alpha|\hat{x}^2|\alpha\rangle - \langle\alpha|\hat{x}|\alpha\rangle^2 \\ &= \frac{\hbar}{2m\omega_0}\left(\alpha^2 + 2|\alpha|^2 + 1 + {\alpha^*}^2 - \alpha^2 - 2|\alpha|^2 - (\alpha^*)^2\right)\end{aligned}$$

$$\boxed{(\Delta x)^2_\alpha = \frac{\hbar}{2mw_0}},$$

which is independent of α (the coherent state shifts the ground state but preserves the shape, and hence its uncertainty).
Displacing the ground state does not change its uncertainty.

Now, we compute the uncertainty for momentum in a coherent state.

$$\begin{aligned}\langle\alpha|\hat{p}|\alpha\rangle &= -i\sqrt{\frac{\hbar m\omega_0}{2}}\langle\alpha|\hat{a} - \hat{a}^\dagger|\alpha\rangle = -i\sqrt{\frac{\hbar m\omega_0}{2}}\,(\alpha - \alpha^*) \\ &= \sqrt{2\hbar m\omega_0}\,\mathrm{Im}\,\alpha \\ \langle\alpha|\hat{p}^2|\alpha\rangle &= -\frac{\hbar m\omega_0}{2}\langle\alpha|\left(\hat{a}^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} - (\hat{a}^\dagger)^2\right)|\alpha\rangle \\ &= -\frac{\hbar m\omega_0}{2}\left(\alpha^2 - 2|\alpha|^2 - 1 - \alpha^{*2}\right)\end{aligned}$$

So

$$\boxed{(\Delta p)^2_\alpha = \frac{\hbar m\omega_0}{2}} \quad \Rightarrow \quad \boxed{(\Delta p)_\alpha(\Delta x)_\alpha = \frac{\hbar}{2}}$$

The uncertainty is unchanged in a coherent state!

4 Time dependence of a coherent state

What about their time dependence? We have not discussed time dependence in general yet, but one should see immediately that

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$

is solved by $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}|\psi(0)\rangle$.
Just check by taking the derivative. Because $\hat{H}$ is independent of time, there is no operator ordering issue. So, we have

$$|\alpha(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}|\alpha\rangle,$$

But $\hat{H} = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$ so

$$\begin{aligned}|\alpha(t)\rangle &= e^{-i\omega_0 t\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)}|\alpha\rangle \\ &= e^{-i\omega_0 t\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)}e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle\end{aligned}$$

How do we proceed? we need to move the two operators through each other. This is the exponential reordering or the braiding identity—we get

$$|\alpha(t)\rangle = \exp\left[e^{-i\omega_0 t\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)}\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)e^{i\omega_0 t\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)}\right]$$

$$\underbrace{e^{-i\omega_0 t\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)}|0\rangle}_{e^{-i\frac{\omega_0 t}{2}}|0\rangle \text{ since } \hat{a}|0\rangle=0}$$

So we need to compute $\hat{U}(t)\hat{a}\hat{U}^\dagger(t)$ and $\hat{U}(t)\hat{a}^+\hat{U}^\dagger(t)$

The easiest way to do this is by differentiating (note this does not always work, but does so here because of the simplicity of $[\hat{H},\hat{a}]$ and $[\hat{H},\hat{a}^\dagger]$). We have

$$\begin{aligned}\frac{d}{dt}\left(\hat{U}(t)\hat{a}\hat{U}^\dagger(t)\right) &= \left(\frac{d}{dt}\hat{U}(t)\right)\hat{a}\hat{U}^\dagger(t) + \hat{U}(t)\hat{a}\left(\frac{d}{dt}\hat{U}^\dagger(t)\right)\\ &= -i\omega_0\hat{U}(t)\left(\hat{a}^\dagger\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a}\right)\hat{U}^\dagger(t)\\ &= -i\omega_0\hat{U}(t)\left[\hat{a}^\dagger\hat{a},\hat{a}\right]\hat{U}^\dagger(t)\\ &= i\omega_0\hat{U}(t)\hat{a}\hat{U}^\dagger(t)\\ &\Rightarrow \boxed{\hat{U}(t)\hat{a}\hat{U}^\dagger(t) = e^{i\omega_0 t}\hat{a}}\end{aligned}$$

similarly

$$\boxed{\hat{U}(t)\hat{a}^\dagger\hat{U}^\dagger(t) = e^{-i\omega_0 t}\hat{a}^\dagger}$$

because of the sign change in the commutator.

Hence

$$\begin{aligned}|\alpha(t)\rangle &= \exp\left[\alpha e^{-i\omega_0 t}\hat{a}^\dagger - \alpha^* e^{i\omega_0 t}\hat{a}\right]e^{-i\frac{\omega_0 t}{2}}|0\rangle\\ &= e^{-i\frac{\omega_0 t}{2}}\left|\alpha e^{-i\omega_0 t}\right\rangle\end{aligned}$$

This implies that the coherent state evolves in time simply via its parameter α being multiplied by $e^{-i\omega t}$ and being multiplied by an overall phase! This is very simple.

5 Relation to classical physics

Let's try to understand by computing the expectation value of position as a function at time

$$\begin{aligned}\langle\alpha(t)|\hat{x}|\alpha(t)\rangle &= \sqrt{\frac{\hbar}{2m\omega_0}}\left\langle\alpha e^{-i\omega_0 t}\right|\left(\hat{a}+\hat{a}^\dagger\right)\left|\alpha e^{-i\omega_0 t}\right\rangle\\ &= \sqrt{\frac{\hbar}{2m\omega_0}}\left(\alpha e^{-i\omega_0 t} + \alpha^* e^{+i\omega_0 t}\right)\\ &= \sqrt{\frac{2\hbar}{m\omega_0}}\left(\mathrm{Re}\alpha\cos\omega_0 t + \mathrm{Im}\,\alpha\sin\omega_0 t\right)\\ &\boxed{\langle\alpha(t)|\hat{x}|\alpha(t)\rangle = x_0\cos\omega_0 t + \frac{p_0}{m\omega_0}\sin\omega_0 t}\end{aligned}$$

This is the classical equation of motion for a spring!

Similarly

$$\begin{aligned}
\langle\alpha(t)|\hat{p}|\alpha(t)\rangle &= -i\sqrt{\frac{\hbar m\omega_0}{2}}\langle\alpha(t)|(\hat{a}-\hat{a}^\dagger)|\alpha(t)\rangle \\
&= -i\sqrt{\frac{\hbar m\omega_0}{2}}\left(\alpha e^{-i\omega_0 t}-\alpha^\dagger e^{i\omega_0 t}\right) \\
&= -i\sqrt{2\hbar m\omega_0}\,(-i\,\mathrm{Re}\,\alpha\sin\omega_0 t + i\,\mathrm{Im}\,\alpha\cos\omega_0 t) \\
&\boxed{\langle\alpha(t)|\hat{p}|\alpha(t)\rangle = -m\omega_0 x_0\sin\omega_0 t + p_0\cos\omega_0 t)}
\end{aligned}$$

Also the classical equation of motion!

You will show on a homework exercise that the uncertainty is independent of time as well.

So we can think of the coherent states as being a "blob" whose uncertainty remains the same for all time and it sloshes back and forth as a classical mass on a spring.

This is as close to a classical image as we get with quantum systems.

6 Energy expectation values in a coherent state

What are the probabilities to observe the system to have different energies when it is prepared in a coherent state? The probability is given by

$$\begin{aligned}
|\langle n|\alpha\rangle|^2 &= \left|\sum_{n=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}e^{-\frac{1}{2}|\alpha|^2}\langle n|m\rangle\right|^2 \\
P(n) &= \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}
\end{aligned}$$

This implies that for any nonzero α, there is a nonzero probability to see any energy excitation.

By differentiating this, we find the maximum occurs when $\frac{d}{d|\alpha|^2}\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2} \Rightarrow \frac{n}{|\alpha|^2} - 1 = 0$ or $|\alpha|^2 = n$. Hence, this maximal probability increases with n. When we later discuss photons, we will see we can think of n as being related to the number of photons and classical sources of light (light bulbs and even lasers) produce light in coherent (and incoherent) states. So no matter how dim the light is, it is never a single photon—there is always the possibility of photon bunching.

An experiment was done in the early days of quantum mechanics of a two slit experiment with light dimmed so low that it took 3 months for enough light to expose the photographic plate. It did show a two slit interference pattern. But this experiment used an incoherent source not a single photon source, so it could not rule out multiple photons being in the apparatus at the same time. Indeed, it is certain that this did occur due to photon bunching. But quantum theory was not well enough developed at that time for anyone to know this was the case, and so the experiment was influential about showing wave-particle duality of photons, even if it had issues.