Phys 506 lecture 5: Squeezed States

The harmonic oscillator is an interesting system particularly because it has a close relationship with photons, which we will explore later. As we saw when we looked at a coherent state, it had the same Δx and Δp of the minimal uncertainty ground state. Can we do better?

Well, we certainly cannot reduce the product of Δx and Δp, but we can trade off the uncertainty—for example, if I find a state where Δx is multiplied by e^{-r} and Δp by e^{r}(r is real), then we can reduce the uncertainty in x at the expense of raising it for p when $r \geq 0$. But if we want to measure x then this may be advantageous. At the very least, it looks like by changing r from $-\infty$ to $+\infty$, we could continuously change from momentum eigenstates to position eigenstates, just by changing a parameter. That would be cool. (Indeed, it does work).

Another exciting thing is working with coherent states gives us more to exercise with our 5 operator identities. Practice makes perfect!

1 Squeezed states

The operator for squeezing can be thought of as a generalization of the displacement operator from being a linear function of $\hat{a}$ and $\hat{a}^\dagger$ to a quadratic one.

We use $\hat{S}(\xi,\eta) = \exp\left[-\frac{\xi}{2}(\hat{a}^\dagger)^2 + \frac{in}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) + \frac{\xi^*}{2}\hat{a}^2\right]$ where ξ can be complex, but η is always real. Why the odd looking choices of parameters? We want $\hat{S}$ to be unitary. But

$$(\hat{S}(\xi,\eta))^\dagger = \exp\left[-\frac{\xi^*}{2}\hat{a}^2 - \frac{i\eta}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) + \frac{\xi}{2}(\hat{a}^\dagger)^2\right]$$
$$= \hat{S}(-\xi,-\eta) = \exp\left[-\left(-\frac{\xi}{2}(\hat{a}^\dagger)^2 + \frac{i\eta}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) + \frac{\xi^*}{2}\hat{a}^2\right)\right].$$

Now, we recall $\exp(\hat{A})\exp(-\hat{A}) = 1$, for any operator $\hat{A}$. Hence

$$(\hat{S}(\xi,\eta))^\dagger = (\hat{S}(\xi,\eta))^{-1}.$$

That implies $\hat{S}$ is unitary!

Since $\hat{S}$ has a quadratic in the exponent, it is like we are controlling the kinetic energy and the potential energy. For example, if we make the potential more confining and narrow, the wave function should be squeezed closer to the origin. This is a way to think about the procedure.

2 Squeezed operators

Let's think of $\hat{S}(\xi,\eta)$ as being a unitary transformation. Then all operators are transformed like $\hat{O} \to \hat{U}^\dagger \hat{O} \hat{U}$ or

$$\hat{a} \to \hat{S}^\dagger(\xi,\eta)\hat{a}\hat{S}(\xi,\eta) = \underbrace{\hat{S}(-\xi,-\eta)\hat{a}\hat{S}(\xi,\eta)}_{\text{Hadamard!}}$$

So

$$\hat{a} \to \hat{a} - [(-\frac{\xi}{2}(\hat{a}^\dagger)^2 + \frac{i\eta}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) + \frac{\xi^*}{2}\hat{a}^2), \hat{a}]$$

$$+\frac{1}{2}[(-\frac{\xi}{2}(\hat{a}^\dagger)^2 + \frac{i\eta}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) + \frac{\xi^*}{2}\hat{a}^2), [(-\frac{\xi}{2}(\hat{a}^\dagger)^2 + \frac{i\eta}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) + \frac{\xi^*}{2}\hat{a}^2), \hat{a}]] + \cdots$$

$$= \hat{a} - (\xi\hat{a}^\dagger - i\eta\hat{a}) + \frac{1}{2}(\xi(\xi^*\hat{a} + i\eta\hat{a}^\dagger) - i\eta(\xi\hat{a}^\dagger - i\eta\hat{a})) + \cdots$$

$$= \hat{a} - (\xi\hat{a}^\dagger - i\eta\hat{a}) + \frac{1}{2}(|\xi|^2 - \eta^2)\hat{a} + \cdots$$

since the $\hat{a}^\dagger$ term vanishes. This actually allows us to perform the infinite sum.

$$\hat{a} \to \hat{a}\left(1 + \frac{1}{2}\left(|\xi|^2 - \eta^2\right) + \frac{1}{4!}|\xi|^2 - \eta^2\right)^2 + \cdots\Bigg)$$

$$- \left(\xi\hat{a}^\dagger - i\eta\hat{a}\right)\left(1 + \frac{1}{3!}\left(|\xi|^2 - \eta^2\right) + \frac{1}{5!}\left(|\xi|^2 - \eta^2\right)^2 + \cdots\right).$$

Now use the hyperbolic trig functions

$$\cosh y = \sum_{m=0}^{\infty} \frac{y^{2m}}{(2m)!} \quad \text{and} \quad \sinh y = \sum_{m=0}^{\infty} \frac{y^{2m+1}}{(2m+1)!},$$

$$\hat{a} \to \hat{a}\cosh\sqrt{|\xi|^2 - \eta^2} - \frac{\left(\eta\hat{a}^\dagger - i\eta\hat{a}\right)}{\sqrt{|\xi|^2 - \eta^2}}\sinh\sqrt{|\xi|)^2 - \eta^2}$$

Hence

$$\boxed{\begin{aligned}\hat{S}^\dagger(\xi,\eta)\hat{a}\hat{S}(\xi,\eta) = \hat{a}&\left[\cosh\sqrt{|\xi|^2 - \eta^2} + \frac{i\eta}{\sqrt{|\xi|^2 - \eta^2}}\sinh\sqrt{|\xi|^2 - \eta^2}\right] \\ &- \hat{a}^\dagger\frac{\xi}{\sqrt{|\xi|^2 - \eta^2}}\sinh\sqrt{|\xi|^2 - \eta^2}\end{aligned}}$$

Taking the Hermitian conjugate gives us

$$\boxed{\begin{aligned}\hat{S}^\dagger(\xi,\eta)\hat{a}^\dagger\hat{S}(\xi,\eta) =& - \hat{a}\frac{\xi^*}{\sqrt{|\xi|^2 - \eta^2}}\sinh\sqrt{|\xi|^2 - \eta^2} \\ &+ \hat{a}^\dagger\left[\cosh\sqrt{|\xi|^2 - \eta^2} - \frac{i\eta^*}{\sqrt{|\xi|^2 - \eta^2}}\sinh\sqrt{|\xi|^2 - \eta^2}\right]\end{aligned}}$$

You may want to look back at what we derived for similar expressions with Pauli matrices. These relations resemble those, but are <u>not</u> identical.

3 Squeezed states and uncertainty

We define $\hat{S}(\xi,\eta)|0\rangle = |\xi,\eta\rangle =$ "squeezed vacuum state".
Then $\langle\xi,\eta|\hat{x}|\xi,\eta\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\langle 0|\hat{S}^\dagger(\xi,\eta)\left(\hat{a}+\hat{a}^\dagger\right)\hat{S}(\xi,\eta)|0\rangle$
Let

$$\boxed{\kappa = \cosh\sqrt{(\xi)^2-\eta^2} + \frac{i\eta}{\sqrt{|\xi|^2-\eta^2}}\sinh\sqrt{|\xi|^2-\eta^2} \quad \lambda = \frac{\xi}{\sqrt{(\xi)^2\eta^2}}\sinh\sqrt{|\xi|^2-\eta^2}}$$

Then $\hat{S}^\dagger\hat{a}\hat{S} = \kappa\hat{a} - \lambda\hat{a}^\dagger$ and $\hat{S}\hat{a}^\dagger\hat{S} = -\lambda^*\hat{a} + \kappa^*\hat{a}^\dagger$. So

$$\langle\xi\eta|\hat{x}|\xi\eta\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\langle 0|(\kappa\hat{a} - \lambda\hat{a}^\dagger - \lambda^*\hat{a} + \kappa^*\hat{a}^\dagger)|0\rangle = 0,$$

since $\hat{a}|0\rangle = 0$ and $\langle 0|\hat{a}^\dagger = 0$.

$$\langle\xi\eta|\hat{x}^2|\xi\eta\rangle = \frac{\hbar}{2m\omega_0}\langle 0|\big((\kappa-\lambda^*)^2\hat{a}^2 + (\kappa-\lambda^*)(\kappa^*-\lambda)\hat{a}\hat{a}^\dagger$$
$$+ (\kappa^*-\lambda)(\kappa-\lambda^*)\hat{a}\hat{a}^\dagger + (\kappa^*-\lambda)^2(a^\dagger)^2\big)|0\rangle$$
$$= \frac{\hbar}{2m\omega_0}(\kappa-\lambda^*)(\kappa^*-\lambda) = \frac{\hbar}{2m\omega_0}(|\kappa|^2 - 2\,\mathrm{Re}\,\kappa\lambda + |\lambda|^2)$$
$$\langle\xi\eta|\hat{p}|\xi\eta\rangle = \sqrt{\frac{\hbar m\omega_0}{2}}\langle 0|\big(\kappa\hat{a} - \lambda\hat{a}^\dagger + \lambda^*\hat{a} - \kappa^*\hat{a}^\dagger\big)|0\rangle = 0$$
$$\langle\xi\eta|\hat{p}^2|\xi\eta\rangle = -\frac{\hbar m\omega_0}{2}\langle 0|\big((\kappa+\lambda^*)^2\hat{a}^2 - (\kappa+\lambda^*)(\kappa^*+\lambda)(\hat{a}a^\dagger + \hat{a}^\dagger\hat{a})$$
$$+ (\kappa^*+\lambda)^2(\hat{a}^\dagger)^2\big)|0\rangle$$
$$= \frac{\hbar m\omega_0}{2}(\kappa+\lambda^*)(\kappa^*+\lambda) = \frac{\hbar m\omega_0}{2}(|\kappa|^2 + 2\,\mathrm{Re}\,\kappa\lambda + |\lambda|^2)$$

Examine for a special case of $\eta = 0$, $\xi = re^{i\phi}$. Then $\kappa = \cosh r$ and $\lambda = \sinh re^{i\phi}$, so

$$(\Delta x)^2_{re^{i\phi},0} = \frac{\hbar}{2m\omega_0}\left(\cosh^2 r - 2\cosh r\sinh r\cos\phi + \sinh^2 r\right)$$
$$= \frac{\hbar}{2m\omega_0}(\cosh 2r - \sinh 2r\cos\phi)$$
$$(\Delta p)^2_{re^{i\phi},0} = \frac{\hbar m\omega_0}{2}(\cosh 2r + \sinh 2r\cos\phi).$$

Let $\phi = 0$, then

$$\cosh 2r + \sinh 2r = e^{2r} \quad \text{and} \quad \cosh 2r - \sinh 2r = e^{-2r}$$

and Δx is squeezed by e^{-r} while Δp is expanded by e^r.
Change $\phi \to \pi$ and it is reversed!

The product of the uncertainties are unchanged, but we have a trade off from x to p and vice versa.

You will explore this more thoroughly on the homework.

4 Expanding squeezed states in terms of energy eigenstates

The last topic we will cover is how to determine the squeezed states themselves. To do this, we need to recall the work from HW #1 on the symplectic group. We saw that

$$\left[\hat{K}_0, \hat{K}_\pm\right] = \pm\hat{K}_\pm \text{ and } \left[\hat{K}_+, \hat{K}_-\right] = -2\hat{K}_0.$$

Here, we claim $\hat{K}_+ = \frac{1}{2}(\hat{a}^\dagger)^2, \quad \hat{K}_- = \frac{1}{2}\hat{a}^2$, and $\hat{K}_0 = \frac{1}{4}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)$.

Check:

$$\begin{aligned}
\left[\hat{K}_+, \hat{K}_-\right] &= \frac{1}{4}\left[(\hat{a}^\dagger)^2, \hat{a}^2\right] \\
&= \frac{1}{4}\left[\hat{a}^\dagger\left[\hat{a}^\dagger, \hat{a}^2\right] + \left[\hat{a}^\dagger, \hat{a}^2\right]\hat{a}^\dagger\right) \\
&= \frac{1}{4}\left(\hat{a}^\dagger\left(\hat{a}\left[\hat{a}^\dagger, \hat{a}\right] + \left[\hat{a}^\dagger, \hat{a}\right)\hat{a}\right) + \left(\hat{a}\left[\hat{a}^\dagger, \hat{a}\right] + \left[\hat{a}^\dagger, \hat{a}\right]\hat{a}\right)\hat{a}^\dagger\right) \\
&= -\frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) \checkmark \\
\left[\hat{K}_0, \hat{k}_+\right] &= \left[\frac{1}{4}\left[\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right), \frac{1}{2}(\hat{a}^\dagger)^2\right] = \frac{1}{8}\left(\hat{a}^\dagger\left[\hat{a}, (\hat{a}^\dagger)^2\right] + \left[\hat{a}, (\hat{a}^\dagger)^2\right]\hat{a}^\dagger\right) \\
&= \frac{1}{8}\left(2\hat{a}^\dagger\hat{a}^\dagger + 2\hat{a}^\dagger\hat{a}^\dagger\right) = \frac{1}{2}(\hat{a}^\dagger)^2 \checkmark \\
\left[\hat{K}_0, \hat{K}_-\right] &= \left[\frac{1}{4}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger, \frac{1}{2}\hat{a}^2\right] = -\frac{1}{2}\hat{a}^2 \checkmark
\end{aligned}$$

So we can immediately use the result from the exponential disentangling identity for the symplectic group

$$\exp\left[-\xi\hat{K}_+ + 2i\eta\hat{K}_0 + \xi^*\hat{K}_-\right] = e^{-\frac{\lambda}{\kappa^*}\hat{K}_+} e^{-2\ln\kappa^*\hat{K}_0} e^{\frac{\lambda^*}{\kappa^*}\hat{K}_-}$$

So, for us, we have $\hat{S}(\xi, \eta) = e^{-\frac{\lambda}{2k\varkappa}(\hat{a}^\dagger)^2} e^{-2\ln\kappa^*\frac{1}{4}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)} e^{\frac{\lambda}{2\kappa^*}\hat{a}^2}$

then the squeezed vacuum (for the special case $\xi = re^{i\phi}, \quad \eta = 0, \quad \kappa = \cosh r$, and $\lambda = e^{i\phi}\sinh r$) becomes

$$S\left(re^{i\phi}, 0\right)|0\rangle = e^{-\frac{e^{i\phi}}{2}\tanh r\left(\hat{a}^\dagger\right)^2} e^{-\frac{2}{4}\ln(\cosh r)\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)} \underbrace{e^{\frac{e^{i\phi}}{2}\tanh(\hat{a})^2}}_{\text{goes to 1}}|0\rangle$$

$$\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)|0\rangle = 1\cdot|0\rangle = |0\rangle$$

$$\boxed{S\left(re^{i\phi}, 0\right)|0\rangle = \frac{1}{\sqrt{\cosh r}}\exp\left(-\tfrac{1}{2}e^{i\phi}\tanh r(\hat{a}^\dagger)^2\right)}$$

Hence, the normalized squeezed vacuum is

$$\boxed{S(re^{i\phi}, 0)|0\rangle = \frac{1}{\sqrt{\cosh r}}\sum_{m=0}^{\infty}\left(-\frac{1}{2}e^{i\phi}\tanh r\right)^m \frac{\sqrt{(2m)!}}{m!}|2m\rangle}$$

One can also look at the displaced squeezed state (note one can squeeze then displace or *vice versa*):

$$\hat{D}(\alpha)\hat{S}(\xi,\eta)|0\rangle$$

or

$$\hat{S}\left(\xi',n'\right)\hat{D}(\alpha')|0\rangle.$$

These states are NOT, in general, equal to each other, but we can find the mapping

$$\xi \to \xi' \quad \eta \to \eta' \text{ and } \alpha \to \alpha'$$

that corresponds to the same state. It is likely to be messy.

Note finally that time dependence of these states is simple due to braiding, so

$$e^{-i\frac{\hat{H}t}{\hbar}}\hat{S}(\xi,\eta)|0\rangle = e^{-i\frac{\omega_0 t}{2}}\hat{S}\left(\xi e^{-\alpha i\omega t},\eta\right)|0\rangle.$$

We have ξ changes periodically with time, but η does not, because $[\hat{H},\hat{a}^\dagger\hat{a}] = [\hat{H},\hat{a}\hat{a}^\dagger] = 0$. So those operators are constants of the motion. This implies η remains constant!

One interesting question is how do we create coherent and squeezed states? In general, it is not so simple. For light, we will find all classical sources of light are coherent states. Squeezing light takes a fair amount of work involving nonlinear optics. This is true about other systems as well. One has to work with a strategy to make such states. It is not so simple (of course the same is true for energy eigenstates $\cdots$).

We can also squeeze and displace excited states, but that ends up not being very useful.

The squeezed vacuum plays an important role in improving the accuracy of LIGO, as we will see when we discuss it later in the class. It can make a significant improvement in the accuracy of the measurements.