Phys 506 lecture 7: Rotations and Angular Momentum

1 Vector operator

In quantum mechanics, we define a vector operator via

$$\left[\hat{v}_i, \hat{L}_j\right] = i\hbar \sum_k \varepsilon_{ijk} \hat{v}_k$$

Oddly, it also satisfies $\left[\hat{L}_i, \hat{v}_j\right] = i\hbar \sum_k \varepsilon_{ijk} \hat{v}_k$ as well. One might have thought there is a sign change, but there isn't. On the homework, you will verify that $\hat{\vec{r}}$ and $\hat{\vec{p}}$ are both vector operators.

2 Rotations

Let's start with rotations. First note that rotations form a group (closed under multiplication, unique inverse, etc.). But they are a non-Abelian group, because when we apply the rotations in one order, they are not the same as when we apply them in the opposite order:

The claim is all rotations can be written as

$$\hat{R}\left(\alpha, \vec{e}_n\right) = \exp\left[-i\frac{\alpha}{\hbar}\vec{e}_n \cdot \hat{\vec{L}}\right]$$

$\vec{e}_n$, a unit vector, is the axis of the rotation and α is the angle of the rotation in radians. $\hat{\vec{L}}$ is the quantum operator for angular momentum. If you look back to what we did with the spin matrices in lecture 1, we showed this holds there as well with $\hat{\vec{L}} \to \hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma}$. (The factor of $\frac{1}{2}$ is needed to be correct).

3 Change of basis

Suppose we transform states via $|\psi\rangle \to \hat{R}|\psi\rangle$, then $\hat{O}|\psi\rangle \to \hat{R}\hat{O}|\psi\rangle = \hat{R}\hat{O}\hat{R}^\dagger(\hat{R}|\psi\rangle)$. This implies that operators transform as $\hat{O} \to \hat{R}\hat{O}\hat{R}^\dagger$.

On the HW, we showed that

$$\hat{R}\left(\alpha, \hat{e}_z\right)\hat{L}_z\hat{R}^\dagger\left(\alpha, \hat{e}_z\right) = \hat{L}_z$$
$$\hat{R}\left(\alpha, \hat{e}_z\right)\hat{L}_x\hat{R}^\dagger\left(\alpha, \hat{e}_z\right) = \cos\alpha\hat{L}_x + \sin\alpha\hat{L}_y$$
$$\hat{R}\left(\alpha, \vec{e}_z\right)\hat{L}_y\hat{R}^\dagger\left(\alpha, \hat{e}_z\right) = -\sin\alpha\hat{L}_x + \cos\alpha\hat{L}_y$$

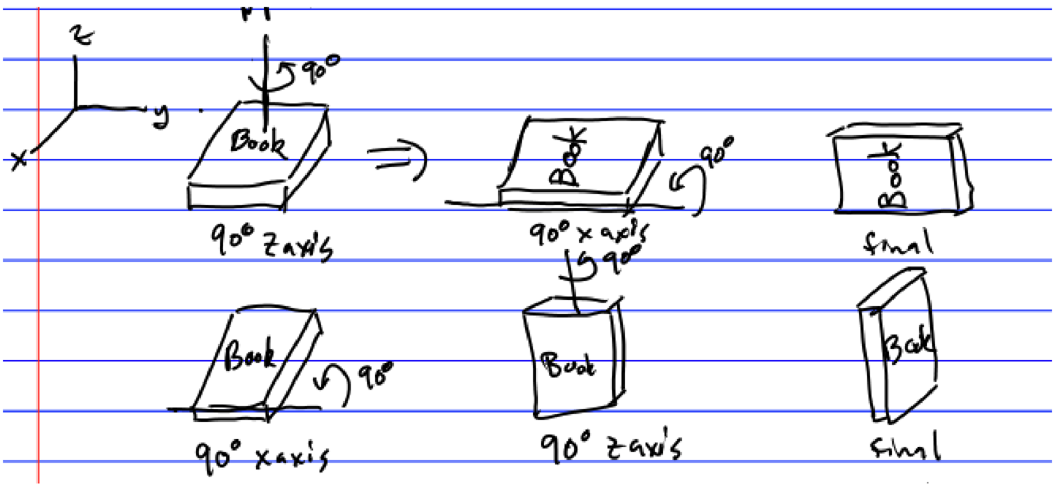

Figure 1: Example of how rotations are not commutative. We start with a book flat on the table in front of us. We rotate by 90 degrees about the vertical axis, and then rotate by 90 degrees along the x-axis. This leaves the book with the front facing towards us and the back spine flat on the table. Reversing the order, we first rotate about the x-axis, where the book stands up with the front facing us, and then 90 degrees about the vertical axis, where the book spine points at us. Each rotation is in a counter-clockwise direction. The final orientations are different, illustrating that rotations do not commute.

We did this by using the Hadamard identity. This is an easier way to do the calculation than to use disentangling. Since a vector operator satisfies the same commutation with $\hat{\vec{L}}$ as $\hat{\vec{L}}$ does (i.e. $\hat{\vec{L}}$ is a vector operator), we have

$$e^{-i\alpha\frac{\hat{L}_z}{\hbar}}\hat{v}_i e^{i\alpha\frac{\hat{L}_z}{\hbar}} = \sum_j O^{(z)}_{ij}\hat{v}_j$$

$$O^{(z)} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Next, via permutations, we can verify that

$$e^{-i\alpha\frac{\hat{L}_x}{\hbar}}\hat{v}_i e^{i\alpha\frac{\hat{L}_x}{\hbar}} = \sum_j O^{(x)}_{ij}\hat{v}_j$$

$$e^{-i\alpha\frac{\hat{L}_y}{\hbar}}\hat{v}_i e^{i\alpha\frac{\hat{L}_y}{\hbar}} = \sum_j O^{(y)}_{ij}\hat{v}_j$$

$$\hat{O}^{(x)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \quad \hat{O}^{(y)} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

What about the general case $e^{-i\alpha\frac{\vec{e}_n\cdot\hat{\vec{L}}}{\hbar}}$? We can work this out with no extra calculation using only geometry. Assume $\vec{e}_n$ points in the θ, ϕ direction.

The spherical unit vectors are

$$\vec{e}_n = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$\vec{e}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$
$$\vec{e}_\phi = (-\sin\phi, \cos\phi, 0)$$

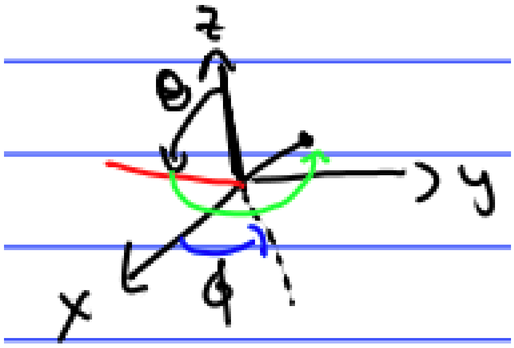

Figure 2: Illustration of the two angles of spherical coordinates—the polar angle θ, given by the rotation about the y-axis and the axial angle ϕ, given by the rotation about the z azis (after the first rotation).

We can use the results we already calculated, assuming we can establish they hold in the new n, θ, ϕ coordinate system. That is, if $\beta \in \{n, \theta, \phi\}$, we need to show

$$\left[\hat{L}_\beta, \hat{v}_\gamma\right] = i\hbar \sum_\delta \varepsilon_{\beta\gamma\delta}\hat{v}_\delta$$

Consider the orthogonal matrix O that satisfies

$$\begin{aligned} \vec{e}_n &= O\vec{e}_z \\ \vec{e}_\theta &= O\vec{e}_x \\ \vec{e}_\phi &= O\vec{e}_y \end{aligned}$$

From the results we found for $\vec{e}_n, \vec{e}_\theta$ and $\vec{e}_\phi$, we have

$$O = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix},$$

but we really only need the fact that $O^T O = 1$ which implies that O is orthogonal.

Note, we have $v_\beta = \sum_j O_{ij} v_j$ with

$$\begin{array}{ccc} \beta & & i \\ n & \leftrightarrow & z \\ \theta & \leftrightarrow & x \\ \phi & \leftrightarrow & y \end{array}$$

The commutation relation is

$$\left[\hat{v}_\beta, \hat{L}_\gamma\right] = i\hbar O_{ii'} O_{jj'} \varepsilon_{i'j'k'} \underbrace{\hat{v}_{k'}}_{\text{Cartesian basis}}$$

Here we have $\beta \leftrightarrow i \quad \gamma \leftrightarrow j$ and repeated indices are summed over. But,

$$v_{k'} = O^T_{k'\delta} O_{\delta l} \hat{v}_l = i\hbar \varepsilon_{i'j'k'} O_{ii'} O_{jj'} \underbrace{O_{\delta k'} \hat{v}_\delta}_{\text{spherical}}.$$

If $\sum_{i'j'k'} \varepsilon_{i'j'k'} O_{ii'} O_{jj'} O_{kk'} = \underbrace{\varepsilon_{ijk}}_{\text{spherical}}$, then we have $[\hat{v}_\beta, \hat{L}_\gamma] = i\hbar\varepsilon_{\beta\gamma\delta}\hat{v}_\delta$ in the spherical basis.

Check: $i = j = n$ $\quad \underbrace{\varepsilon_{i'j'k'}}_{\text{anti-symmetric on interchange of } i'j'} \quad \underbrace{O_{ni'} O_{kj'}}_{\text{symmetric on interchange of } i'j'} \quad O_{kk'} \Rightarrow O$

$$
\begin{aligned}
&i = n, j = \theta : \\
&\Rightarrow \varepsilon_{xyz} O_{nx} O_{\theta y} O_{\delta z} + \varepsilon_{yzx} O_{ny} O_{\theta z} O_{\delta x} + \varepsilon_{zxy} O_{nz} O_{\theta x} O_{\delta y} \\
&+ \varepsilon_{xzy} O_{nx} O_{\theta z} O_{\delta y} + \varepsilon_{zyx} O_{nz} O_{\theta y} O_{\delta x} + \varepsilon_{yxz} O_{ny} O_{\theta x} O_{\delta z}
\end{aligned}
$$

The top row has all ε's $+1$, bottom has all -1. Careful inspection shows this is $\det O$. But since $O^T O = \mathbb{I} \Rightarrow \det O^T \det O = 1 \Rightarrow \det O = \pm 1$ for rotations, we find $\det O = 1$ (just calculate the determinant of the explicit matrix O above).

4 Angular momentum operators and eigenstates

It is easy to see from this that we have established the result we need. Essentially we have after the rotation $\vec{e}_z \to \vec{e}_n \quad \vec{e}_x \to \vec{e}_\theta \quad \vec{e}_y \to \vec{e}_\phi$.

You should have already verified from the algebras of $\hat{L}_x, \hat{L}_y$ and $\hat{L}_z$ that

$$
\begin{aligned}
\hat{L}_z |lm\rangle &= \hbar m |lm\rangle \\
\hat{L}_+ |lm\rangle &= \hbar\sqrt{l(l+1) - m(m+1)}|lm+1\rangle \\
&= \hbar\sqrt{(l-m)(l+m+1)}|lm+1\rangle \\
\hat{L}_- |lm\rangle &= \hbar\sqrt{l(l+1) - m(m-1)}|lm-1\rangle \\
&= \hbar\sqrt{(l+m)(l-m+1)}|lm-1\rangle \\
\hat{L}^2 |lm\rangle &= \hbar^2 l(l+1)|lm\rangle
\end{aligned}
$$

Using $\hat{L}^2 = \frac{1}{2}\left(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+\right) + \hat{L}_z\hat{L}_z$ and the above results we see that because

$$
\begin{aligned}
&\frac{\hbar^2}{2}[\sqrt{(l-m)(l+m+1)}\sqrt{(l+m+1)(l-m)} \\
&\quad + \sqrt{(l+m)(l-m+1)(l-m+1)(l+m)}] + \hbar^2 m^2 \\
&= \hbar^2\left[\frac{1}{2}(l-m)(l+m+1) + \frac{1}{2}(l+m)(l-m+1) + m^2\right] \\
&= \hbar^2\left[\frac{1}{2}\left(l^2 - m^2 + l - m + l^2 - m^2 + l + m\right) + m^2\right] \\
&= \hbar^2 l(l+1).
\end{aligned}
$$

We can put these results together to verify that

$$|\theta, \phi\rangle = e^{-i\phi\frac{\hat{L}_z}{\hbar}} e^{-i\theta\frac{\hat{L}_y}{\hbar}} |\theta{=}0, \phi{=}0\rangle.$$

As a check, we verify that

$$
\begin{aligned}
e^{i\hat{\theta}}|\theta, \phi\rangle &= e^{i\theta}|\theta, \phi\rangle \\
e^{i\hat{\phi}}|\theta, \phi\rangle &= e^{i\phi}|\theta, \phi\rangle
\end{aligned}
$$

with $e^{i\hat{\theta}} = \frac{\hat{z}+i\sqrt{\hat{x}^2+\hat{y}^2}}{\hat{r}} \quad e^{i\hat{\phi}} = \frac{\hat{x}+i\hat{y}}{\sqrt{\hat{x}^2+\hat{y}^2}}$. We must define these in this way because $\hat{\theta}$ and $\hat{\phi}$ are not well-defined quantum operators.

Because $\hat{r}$ commutes with $\hat{\vec{L}}$ and $(\hat{x}, \hat{y}, \hat{z})$ is a vector operator, we can establish this result via Hadamard. You will verify this on the homework.

5 Exponential disentangling and angular momentum

We end by discussing exponential disentangling. We will need to compute

$$e^{i\theta\frac{\hat{L}_y}{\hbar}} = e^{i\frac{\theta}{\hbar}\frac{\hat{L}_+-\hat{L}_-}{2i}} = e^{\frac{\theta}{2\hbar}\left(\hat{L}_+-\hat{L}_-\right)}$$

We derived disentangling in lecture 1. It gives

$$e^{\frac{\theta}{2\hbar}(\hat{L}_+-\hat{L}_-)} = e^{-\tan\left(\frac{\theta}{2}\right)\frac{\hat{L}_-}{\hbar}-} e^{2\ln\cos\frac{\theta}{2}\frac{\hat{L}_z}{\hbar}} e^{\tan\left(\frac{\theta}{2}\right)\frac{\hat{L}_z}{\hbar}}$$

(there is a similar one with $\hat{L}_-$ on the right and $\hat{L}_+$on the left)

We will use this to find spherical harmonics in the next lecture. In particular, note that

$$\begin{aligned}|\theta, \phi\rangle &= e^{-i\phi\frac{\hat{L}_z}{\hbar}} e^{-i\frac{\theta\hat{L}_y}{\hbar}} |\theta{=}0, \phi{=}0\rangle \\ &= e^{-i\phi\frac{\hat{L}_z}{\hbar}} e^{\tan\left(\frac{\theta}{2}\right)\frac{\hat{L}_-}{\hbar}} e^{2\ln\cos\frac{\theta}{2}\frac{\hat{L}_z}{\hbar}} e^{-\tan\left(\frac{\theta}{2}\right)\frac{\hat{L}_+}{\hbar}} |\theta{=}0, \phi{=}0\rangle\end{aligned}$$

At the moment this does not look too exciting, but we will see that it is exactly what we need in the next lecture.