Phys 506 lecture 9: Central Forces

This is a somewhat technical lecture.

1 Average and relative coordinates

Consider two particles interacting via a potential that depends only on the distance between them, i.e. a central potential:

$$\hat{H} = \frac{\hat{\mathbf{p}}_1^2}{2m_1} + \frac{\hat{\mathbf{p}}_2^2}{2m_2} + V(|\mathbf{r}_1 - \mathbf{r}_2|).$$

It turns out that this problem can be mapped to an effective one-particle problem. We do so by using "average" and "relative" coordinates. The **center of mass coordinate $\hat{\mathbf{R}}$**, which is an operator, is defined via:

$$\hat{\mathbf{R}} = \frac{m_1\hat{\mathbf{r}}_1 + m_2\hat{\mathbf{r}}_2}{m_1 + m_2}.$$

Because the momentum is proportional to each mass, we can try to define the **average momentum coordinate** to be conjugate to the center of mass coordinate as follows:

$$\hat{\mathbf{P}} = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2.$$

Using these definitions for the average coordinates, we find that the usual commutation relations are still satisfied:

$$\begin{aligned}
[\hat{\mathbf{R}}_\alpha, \hat{\mathbf{P}}_\beta] &= \left[\frac{m_1\hat{\mathbf{r}}_{1\alpha} + m_2\hat{\mathbf{r}}_{2\alpha}}{m_1 + m_2}, \hat{\mathbf{p}}_{1\beta} + \hat{\mathbf{p}}_{2\beta}\right] \\
&= \frac{1}{m_1 + m_2}\left(m_1[\hat{\mathbf{r}}_{1\alpha}, \hat{\mathbf{p}}_{1\beta}] + m_2[\hat{\mathbf{r}}_{2\alpha}, \hat{\mathbf{p}}_{2\beta}]\right) \\
&= \frac{1}{m_1 + m_2}(m_1 i\hbar\delta_{\alpha\beta} + m_2 i\hbar\delta_{\alpha\beta}) \\
&= i\hbar\delta_{\alpha\beta} \quad \checkmark.
\end{aligned}$$

Now let's define the **relative position** as:

$$\hat{\mathbf{r}} = \hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2.$$

We use lowercase symbols to represent the relative coordinates.

Since momentum is proportional to mass, if we define our relative momentum as:

$$\hat{\mathbf{p}} = \frac{m_1 m_2}{m_1 + m_2}\left(\frac{\hat{\mathbf{p}}_1}{m_1} - \frac{\hat{\mathbf{p}}_1}{m_2}\right),$$

also with a lower-case letter. We can check that the usual commutation relations are again satisfied:

$$\begin{aligned}[\hat{\mathbf{r}}_\alpha, \hat{\mathbf{p}}_\beta] &= \left[\hat{\mathbf{r}}_{1,\alpha} - \hat{\mathbf{r}}_{2,\alpha}, \frac{m_1 m_2}{m_1+m_2}\left(\frac{\hat{\mathbf{p}}_{1,\beta}}{m_1} - \frac{\hat{\mathbf{p}}_{2,\beta}}{m_2}\right)\right] \\ &= \frac{m_1 m_2}{m_1+m_2}\left(\frac{1}{m_1}[\hat{\mathbf{r}}_{1,\alpha}, \hat{\mathbf{p}}_{1,\beta}] + \frac{1}{m_2}[\hat{\mathbf{r}}_{2,\alpha}, \hat{\mathbf{p}}_{2,\beta}]\right) \\ &= \frac{m_1 m_2}{m_1+m_2}\left(\frac{1}{m_1} i\hbar\delta_{\alpha\beta} + \frac{1}{m_2} i\hbar\delta_{\alpha\beta}\right) \\ &= i\hbar\delta_{\alpha\beta} \quad \checkmark .\end{aligned}$$

Furthermore, it is easy to check that relative and average operators commute:

$$[\hat{\mathbf{R}}_\alpha, \hat{\mathbf{p}}_\beta] = [\hat{\mathbf{r}}_\alpha, \hat{\mathbf{P}}_\beta] = 0.$$

Note further that we can write

$$\begin{aligned}\hat{\mathbf{p}}_1 &= \left(\hat{\mathbf{P}} \;+\; \frac{m_1+m_2}{m_1\,m_2}\,\hat{\mathbf{p}}\right)\frac{m_1}{m_1+m_2}, \\ \hat{\mathbf{p}}_2 &= \left(\hat{\mathbf{P}} \;-\; \frac{m_1+m_2}{m_1\,m_2}\,\hat{\mathbf{p}}\right)\frac{m_2}{m_1+m_2},\end{aligned}$$

to express the original momentum operators in terms of the average and relative ones. Hence, we can show:

$$\begin{aligned}\frac{\hat{\mathbf{p}}_1^2}{2\,m_1} \;+\; \frac{\hat{\mathbf{p}}_2^2}{2\,m_2} &= \frac{1}{2}\,\frac{m_1}{(m_1+m_2)^2}\left(\hat{\mathbf{P}}^2 \;-\; 2\,\frac{m_1+m_2}{m_1}\,\hat{\mathbf{P}}\cdot\hat{\mathbf{p}} \;+\; \left(\frac{m_1+m_2}{m_1}\right)^2\hat{\mathbf{p}}^2\right) \\ &\quad + \frac{1}{2}\,\frac{m_2}{(m_1+m_2)^2}\left(\hat{\mathbf{P}}^2 - 2\,\frac{m_1+m_2}{m_2}\,\hat{\mathbf{P}}\cdot\hat{\mathbf{p}} + \left(\frac{m_1+m_2}{m_2}\right)^2\hat{\mathbf{p}}^2\right) \\ &= \frac{1}{2}\,\frac{\hat{\mathbf{P}}^2}{(m_1+m_2)} + \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\hat{\mathbf{p}}^2\end{aligned}$$

We can simplify this expression by defining the term $\mu = \frac{m_1 m_2}{m_1+m_2}$ as the reduced mass. As a result, the kinetic energy term in the Hamiltonian becomes:

$$\frac{\hat{\mathbf{p}}_1^2}{2\,m_1} + \frac{\hat{\mathbf{p}}_2^2}{2\,m_2} = \frac{\hat{\mathbf{P}}^2}{2\,(m_1+m_2)} + \frac{\hat{\mathbf{p}}^2}{2\,\mu}$$

And the full Hamiltonian becomes:

$$\boxed{H = \underbrace{\frac{1}{2}\,\frac{\hat{\mathbf{P}}^2}{(m_1+m_2)}}_{\text{center of mass}} + \underbrace{\frac{1}{2}\,\frac{\hat{\mathbf{P}}^2}{\mu} + V\big(|\hat{\mathbf{r}}|\big)}_{\text{relative motion}}}$$

We have thus split the Hamiltonian into a center of mass motion part, represented by the term $\frac{1}{2}\frac{\hat{\mathbf{P}}^2}{(m_1+m_2)}$, which acts like an effective free particle, and a relative motion term, given by the term $\frac{1}{2}\frac{\hat{\mathbf{p}}^2}{\mu} + V\big(|\hat{\mathbf{r}}|\big)$, which acts like an effective particle in a potential.

Going forward, we focus on the relative part. Our objective now is to rewrite the kinetic energy in terms of the radial and angular contributions. We will start with the radial momentum.

2 Radial momentum

We define the **radial momentum** as:

$$\hat{p}_r = \frac{1}{2}\left(\frac{\mathbf{r}}{r}\cdot\hat{\mathbf{p}} + \hat{\mathbf{p}}\cdot\frac{\mathbf{r}}{r}\right)$$

where we have used a symmetric form for the combination of terms because it is guaranteed to be Hermitian. Note that $\hat{\mathbf{r}}/\hat{r}$ is an operator unit vector in the radial direction.

Using the two identities:

$$[\hat{p}_\alpha, \hat{r}] = -i\hbar\,\frac{\hat{r}_\alpha}{\hat{r}},$$
$$\left[\hat{p}_\alpha, \frac{1}{r}\right] = i\hbar\,\frac{\hat{r}}{\hat{r}^3},$$

which you will prove on the homework, we can show:

$$\hat{p}_r = \frac{1}{2}\left(\frac{\hat{\mathbf{r}}}{\hat{r}}\cdot\hat{\mathbf{p}} + \frac{\hat{\mathbf{r}}}{\hat{r}}\cdot\hat{\mathbf{p}} + \sum_\alpha\left[\hat{\mathbf{p}}_\alpha, \frac{\hat{\mathbf{r}}_\alpha}{\hat{r}}\right]\right),$$
$$= \frac{1}{\hat{r}}\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} + \frac{1}{2}\left(-i\hbar\times 3\frac{1}{\hat{r}} + \sum_\alpha \hat{r}_\alpha i\hbar\frac{\hat{r}_\alpha}{\hat{r}^3},\right)$$

As a result, the radial momentum becomes:

$$\boxed{\hat{p}_r = \frac{1}{\hat{r}}(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}) - \frac{i\hbar}{\hat{r}}.}$$

Next, notice that:

$$[\hat{r}, \hat{p}_r] = \left[\hat{r}, \frac{1}{\hat{r}}\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - \frac{i\hbar}{\hat{r}}\right] = \frac{1}{\hat{r}}\sum_\alpha \hat{r}_\alpha[\hat{r}, \hat{p}_\alpha] = \frac{1}{\hat{r}}\sum_\alpha \hat{r}_\alpha i\hbar\frac{\hat{r}_\alpha}{\hat{r}} = i\hbar$$

Hence, we can think of $\hat{r}$ and $\hat{p}_r$ as canonically conjugate operators. Note that $\hat{p}_r$ is Hermitian but is not self-adjoint, so it has no complete set of eigenstates. However, this does not affect anything we do. It mainly means we should not expand in terms of eigenstates of the radial momentum—we never do that.

3 Separation of variables

We show how to separate the kinetic energy into its radial and angular components.

We first want to compute the radial contribution:

$$\hat{p}_r^2 = \frac{1}{\hat{r}}(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - i\hbar)\frac{1}{\hat{r}}(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - i\hbar)$$

To do so, we can note that since $[\hat{r}, \hat{p}_r] = i\hbar$, we have:

$$\left[\frac{\hat{r}}{\hat{r}}, \hat{p}_r\right] = \frac{1}{\hat{r}}\,[\hat{r}, \hat{p}_r] + \left[\frac{1}{\hat{r}}, \hat{p}_r\right]\hat{r} = 0$$

which implies:

$$\left[\frac{1}{\hat{r}}, \hat{p}_r\right] = -\frac{i\hbar}{\hat{r}^2}.$$

Using this, we get:

$$\begin{aligned}
\hat{p}_r^2 &= \left(\frac{1}{\hat{r}^2}\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - i\hbar\right) + \left[\hat{p}_r, \frac{1}{\hat{r}}\right]\right)\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - i\hbar\right) \\
&= \frac{1}{\hat{r}^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - i\hbar\right) \\
&= \frac{1}{\hat{r}^2}\sum_{\alpha,\beta}\hat{r}_\alpha\hat{p}_\alpha\left(\hat{r}_\beta\hat{p}_\beta - i\hbar\right) \\
&= \frac{1}{\hat{r}^2}\sum_{\alpha\beta}\left(\hat{r}_\alpha\hat{r}_\beta\hat{p}_\alpha\hat{p}_\beta - \hat{r}_\alpha\hat{p}_\beta i\hbar\delta_{\alpha\beta} - i\hbar\hat{r}_\alpha\hat{p}_\alpha\right) \\
&= \frac{1}{\hat{r}^2}\sum_{\alpha\beta}\left(\hat{r}_\alpha\hat{r}_\beta\hat{p}_\beta\hat{p}_\alpha - 2i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}\right) \\
&= \frac{1}{\hat{r}^2}\Big[\hat{r}_x^2\hat{p}_x^2 + \hat{r}_y^2\hat{p}_y^2 + \hat{r}_z^2\hat{p}_z^2 + 2\hat{r}_x\hat{r}_y\hat{p}_x\hat{p}_y + 2\hat{r}_x\hat{r}_z\hat{p}_x\hat{p}_z + 2\hat{r}_y\hat{r}_z\hat{p}_y\hat{p}_z \\
&\quad - 2i\hbar(\hat{r}_x\hat{p}_x + \hat{r}_y\hat{p}_y + \hat{r}_z\hat{p}_z)\Big].
\end{aligned}$$

Next, we have the angular contribution, which comes from

$$\begin{aligned}
(\hat{\mathbf{r}}\times\hat{\mathbf{p}})^2 &= (\hat{r}_x\hat{p}_y - \hat{r}_y\hat{p}_x)(\hat{r}_x\hat{p}_y - \hat{r}_y\hat{p}_x) + (\hat{r}_y\hat{p}_z - \hat{r}_z\hat{p}_y)(\hat{r}_y\hat{p}_z - \hat{r}_z\hat{p}_y) \\
&\quad + (\hat{r}_z\hat{p}_x - \hat{r}_x\hat{p}_z)(\hat{r}_z\hat{p}_x - \hat{r}_x\hat{p}_z) \\
&= \hat{r}_x^2\hat{p}_y^2 + \hat{r}_y^2\hat{p}_x^2 - 2\hat{r}_x\hat{r}_y\hat{p}_x\hat{p}_y + i\hbar\hat{r}_y\hat{p}_y + i\hbar\hat{r}_x\hat{p}_x \\
&\quad + \hat{r}_y^2\hat{p}_z^2 + \hat{r}_z^2\hat{p}_y^2 - 2\hat{r}_y\hat{r}_z\hat{p}_y\hat{p}_z + i\hbar\hat{r}_z\hat{p}_z + i\hbar\hat{r}_y\hat{p}_y \\
&\quad + \hat{r}_z^2\hat{p}_x^2 + \hat{r}_x^2\hat{p}_z^2 - 2\hat{r}_z\hat{r}_x\hat{p}_z\hat{p}_x + i\hbar\hat{r}_x\hat{p}_x + i\hbar\hat{r}_z\hat{p}_z.
\end{aligned}$$

As a result, we obtain:

$$\hat{p}_r^2 + \frac{1}{\hat{r}^2}\hat{\mathbf{L}}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$$

so:

$$\boxed{\hat{H} = \frac{\hat{p}_r^2}{2\mu} + \frac{\hat{\mathbf{L}}^2}{2\mu\hat{r}^2} + V(\hat{r})}$$

We will often write the square of the angular momentum operator as $\hat{L}^2$, similar to what we do for the radial coordinate.

4 Radial translation operator

The last thing we work on is the **radial translation operator**. It is just the translation operator expressed in terms of $\hat{p}_r$, $\hat{p}_\theta$, $\hat{p}_\phi$, $\cos\hat{\theta}$, $\sin\hat{\theta}$, $\cos\hat{\phi}$, and $\sin\hat{\phi}$. Doing so requires a few subtle points. We do not go through the full details, but we describe most of the issues. The translation operator in Cartesian coordinates satisfies

$$e^{-\frac{i}{\hbar}(x\hat{p}_x + y\hat{p}_y + z\hat{p}_z)},$$

with x, y, and z numbers, not operators. Recall that:

$$\hat{p}_r = \frac{1}{\hat{r}}\left(\hat{r}_x\hat{p}_x + \hat{r}_y\hat{p}_y + \hat{r}_z\hat{p}_z - i\hbar\right) = \sin\hat{\theta}\cos\hat{\phi}\,\hat{p}_x + \sin\hat{\theta}\sin\hat{\phi}\,\hat{p}_y + \cos\hat{\theta}\,\hat{p}_z - \frac{i\hbar}{\hat{r}}.$$

We also compute the analogs of $\hat{p}_\theta = \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{p}}$ and $\hat{p}_\phi = \hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{p}}$ similar to how we computed $\hat{p}_r$ (symmetrizing to make them Hermitian):

$$\hat{p}_\theta = \cos\hat{\theta}\cos\hat{\phi}\,\hat{p}_x + \cos\hat{\theta}\sin\hat{\phi}\,\hat{p}_y - \sin\hat{\theta}\,\hat{p}_z - \frac{i\hbar}{\hat{r}}\frac{\cot\hat{\theta}}{\hat{r}},$$

$$\hat{p}_\phi = -\sin\hat{\phi}\,\hat{p}_x + \cos\hat{\phi}\,\hat{p}_y.$$

Note that $\hat{p}_\theta$ requires a quantum correction, but $\hat{p}_\phi$ does not. You will show this on the homework. We use $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ (here it is the x, y, z of the translations and all are numbers).

Our strategy is to show that the translation can be written as a translation along z followed by the rotations by θ along the y-axis and ϕ along the z-axis.

$$\begin{aligned}|x, y, z\rangle &= e^{-i\frac{\phi}{\hbar}\hat{L}_z} e^{-i\frac{\theta}{\hbar}\hat{L}_y} e^{-i\frac{r}{\hbar}\hat{p}_z}|0\rangle \\ &= e^{-i\frac{\phi}{\hbar}\hat{L}_z} e^{-i\frac{\theta}{\hbar}\hat{L}_y} e^{-i\frac{r}{\hbar}\hat{p}_z} e^{-i\frac{\theta}{\hbar}\hat{L}_y} e^{-i\frac{\phi}{\hbar}\hat{L}_z}|0\rangle\end{aligned}$$

Because $\hat{L}_z = \hat{r}_x\hat{p}_y - \hat{r}_y\hat{p}_x$, $\hat{L}_y = \hat{r}_z\hat{p}_x - \hat{r}_x\hat{p}_z$ and $\hat{r}_\alpha|0\rangle = 0$. But:

$$e^{-i\frac{\theta}{\hbar}\hat{L}_y}\hat{p}_z e^{i\frac{\theta}{\hbar}\hat{L}_y} = \sin\theta\hat{p}_x + \cos\theta\hat{p}_z,$$

and:

$$e^{-i\frac{\phi}{\hbar}\hat{L}_z}\hat{p}_x e^{i\frac{\phi}{\hbar}\hat{L}_z} = \cos\phi\hat{p}_x + \sin\phi\hat{p}_y.$$

Therefore:

$$\begin{aligned}|x, y, z\rangle &= e^{-i\frac{r}{\hbar}(\sin\theta\cos\phi\hat{p}_x + \sin\theta\sin\phi\hat{p}_y + \cos\theta\hat{p}_z)}|0\rangle \\ &= e^{-i\frac{1}{\hbar}(x\hat{p}_x + y\hat{p}_y + z\hat{p}_z)}|0\rangle\end{aligned}$$

as expected. Now go back to the original form:

$$|x, y, z\rangle = e^{-\frac{i}{\hbar}\phi\hat{L}_z} e^{-\frac{i}{\hbar}\theta\hat{L}_y} e^{-\frac{i}{\hbar}r\hat{p}_z}|0\rangle.$$

But:

$$\hat{p}_z = \cos\hat{\theta}\,\hat{p}_r - \sin\hat{\theta}\,\hat{p}_\theta + i\hbar\frac{\cos\hat{\theta}}{\hat{r}} - \frac{i\hbar}{\hat{r}^2}\cos\hat{\theta}.$$

This implies

$$|x, y, z\rangle = e^{-i\frac{\phi}{\hbar}\hat{L}_z} e^{-i\frac{\theta}{\hbar}\hat{L}_y} e^{-i\frac{r}{\hbar}\left(\cos\hat{\theta}\,\hat{p}_r - \sin\hat{\theta}\,\hat{p}_\theta + i\frac{\hbar}{2\hat{r}}\cos\hat{\theta}\right)}|0\rangle.$$

Note that because we are translating along the z-axis, we will have $\cos\hat{\theta}|0\rangle = |0\rangle$ and $\sin\hat{\theta}|0\rangle = 0$. Now recall:

$$\hat{p}_r = \frac{1}{\hat{r}}\left(\hat{r}_x\hat{p}_x + \hat{r}_y\hat{p}_y + \hat{r}_z\hat{p}_z\right) - \frac{i\hbar}{\hat{r}},$$

and

$$\left[\frac{\hat{r}_\alpha}{\hat{r}}, \hat{p}_r\right] = \frac{1}{\hat{r}}\left([\hat{r}_\alpha, \hat{p}_r]\right) + \left[\frac{1}{\hat{r}}, \hat{p}_r\right]\hat{r}_\alpha = \frac{1}{\hat{r}^2}\hat{r}_\alpha\, i\hbar + \frac{1}{\hat{r}^2}\hat{r}_\alpha\,(-i\hbar) = 0.$$

which implies that $\hat{p}_r$ commutes with $\cos\hat{\theta}$ and $\sin\hat{\theta}$.

$$\hat{p}_\theta = \cos\hat{\theta}\cos\hat{\phi}\,\hat{p}_x + \cos\hat{\theta}\sin\hat{\phi}\,\hat{p}_y - \sin\hat{\theta}\,\hat{p}_z - \frac{i\hbar}{\hat{r}}\cos\hat{\theta}.$$

Note that:

$$\begin{aligned} -\sin\hat{\phi}\,\hat{L}_x + \cos\hat{\phi}\,\hat{L}_y &= -\sin\hat{\phi}\,\sin\hat{\theta}\,\hat{p}_z + \sin\hat{\phi}\,\cos\hat{\theta}\,\hat{p}_x + \cos\hat{\phi}\,\cos\hat{\theta}\,\hat{p}_y - \cos\hat{\phi}\,\sin\hat{\theta}\,\hat{p}_z \\ &= \cos\hat{\theta}\,\cos\hat{\phi}\,\hat{p}_x + \cos\hat{\theta}\,\sin\hat{\phi}\,\hat{p}_y - \sin\hat{\theta}\,\hat{p}_z. \end{aligned}$$

But:

$$\begin{aligned} \sin\hat{\theta}\,\hat{p}_\theta &= \sin\hat{\theta}\,\cos\hat{\phi}\,\hat{p}_x + \sin\hat{\theta}\,\sin\hat{\phi}\,\hat{p}_y - \sin^2\hat{\theta}\,\hat{p}_z - \frac{i\hbar}{2\hat{r}}\cos\hat{\theta} \\ &= \frac{\sin\hat{\theta}}{\hat{r}}\left(-\sin\hat{\phi}\,\hat{L}_x + \cos\hat{\phi}\,\hat{L}_y\right) - \frac{i\hbar}{2\hat{r}}\cos\hat{\theta} \\ &= \frac{-\hat{r}_y\hat{L}_x + \hat{r}_x\hat{L}_y}{\hat{r}^2} - \frac{i\hbar}{2\hat{r}}\cos\hat{\theta}. \end{aligned}$$

Hence:

$$\begin{aligned} \cos\hat{\theta}\,\hat{p}_r - \sin\hat{\theta}\,\hat{p}_\theta + \frac{i\hbar}{2\hat{r}}\cos\hat{\theta} &= \cos\hat{\theta}\,\hat{p}_r + \frac{\hat{r}_y\hat{L}_x - \hat{r}_x\hat{L}_y}{\hat{r}^2} + \frac{i\hbar}{\hat{r}}\cos\hat{\theta}, \\ &= \left(\hat{p}_r + \frac{i\hbar}{\hat{r}}\right)\cos\hat{\theta} + \frac{\hat{r}_x\sin\hat{\theta} - \hat{r}_y\cos\hat{\theta}}{\hat{r}^2} \\ &= \left(\hat{p}_r + \frac{i\hbar}{\hat{r}}\right)\cos\hat{\theta} + \frac{\hat{r}_x\sin\hat{\theta} - \hat{r}_y\cos\hat{\theta}}{\hat{r}^2} - i\hbar\frac{\hat{r}_z}{\hat{r}^2} + i\hbar\frac{\hat{r}_z}{\hat{r}^2} \\ &= \left(\hat{p}_r - \frac{i\hbar}{\hat{r}}\right)\cos\hat{\theta} + \frac{\hat{L}_x\sin\hat{\theta}\sin\hat{\phi} - \hat{L}_y\sin\hat{\theta}\cos\hat{\phi}}{\hat{r}} \\ &= \left(\hat{p}_r - \frac{i\hbar}{\hat{r}}\right)\cos\hat{\theta} + \hat{p}_z\sin^2\hat{\theta}\,\sin^2\hat{\phi} - \hat{p}_y\cos\hat{\theta}\,\sin^2\hat{\theta} - \hat{p}_x\cos^2\hat{\theta}\,\cos^2\hat{\phi} \\ &\quad + \hat{p}_z\sin^2\hat{\theta}\cos^2\hat{\phi} \\ &= \left(\hat{p}_r - \frac{i\hbar}{\hat{r}}\right)\cos\hat{\theta} - \hat{p}_y\cos\hat{\theta}\,\sin^2\hat{\theta} - \hat{p}_x\cos^2\hat{\theta}\,\cos^2\hat{\phi} + \hat{p}_z\sin^2\hat{\theta}. \end{aligned}$$

When this acts on $|0\rangle$, it gives $\left(\hat{p}_r - \frac{i\hbar}{\hat{r}}\right)|0\rangle$ because $\sin\hat{\theta}\,|0\rangle = 0$. Furthermore, since $\sin\hat{\theta}$ commutes with $\hat{p}_r - \frac{i\hbar}{\hat{r}}$, we find that raising this operator to any power n acting on $|0\rangle$ also gives $\left(\hat{p}_r - \frac{i\hbar}{\hat{r}}\right)^n$ acting on $|0\rangle$. This gives our final result:

$$\boxed{|x,y,z\rangle = e^{-i\frac{\phi}{\hbar}\hat{L}_z}e^{-i\frac{\theta}{\hbar}\hat{L}_y}e^{-i\frac{r}{\hbar}\left(\hat{p}_r - \frac{i\hbar}{\hat{r}}\right)}|0\rangle}$$

This is the **translation operator in spherical coordinates**. Note that it has a form that appears to be non-unitary, but as we saw, this operator does produce the correct position space state. Indeed, the quantum connection term is critical to it giving the right behavior. Furthermore, it only works when operating on the position state at the origin.