Bose coherent states

Suppose I have a vacuum state $|0\rangle$ such that $\hat{a}|0\rangle = 0$ with $\hat{a}$ a bosonic destruction operator $[\hat{a}, \hat{a}^+]_- = 1$

Let $\frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle = |n\rangle$ be the state with n bosons. Consider a superposition of all such states

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n\rangle \qquad \psi_n = \text{number}$$

This state can never be an eigenstate of $\hat{a}^+$.

$$\hat{a}^+|\psi\rangle = \sum_{n=0}^{\infty} \psi_n \hat{a}^+|n\rangle = \sum_{n=0}^{\infty} \psi_n \sqrt{n+1}\,|n+1\rangle$$

$$= \sum_{n=1}^{\infty} \psi_{n-1} \sqrt{n}\,|n\rangle$$

↑ state $|0\rangle$ is not in sum, but it is in $|\psi\rangle$

so $\hat{a}^+|\psi\rangle$ can never equal number $|\psi\rangle$.

But it can be an eigenstate of the destruction operator

$$\hat{a}|\psi\rangle = \sum_{n=0}^{\infty} \psi_n a|n\rangle = \sum_{n=1}^{\infty} \psi_n \sqrt{n}\,|n-1\rangle$$

$$= \sum_{n=0}^{\infty} \psi_{n+1} \sqrt{n+1}\,|n\rangle$$

$\Rightarrow$ if $\psi_n = c\,\psi_{n+1}\sqrt{n+1}$ we have an eigenstate.

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choose $\psi_n = \frac{\phi^n}{\sqrt{n!}}$ to get $c = \phi^{-1}$

or $\hat{a}\,|\psi\rangle = \hat{a} \sum_{n=0}^{\infty} \frac{\phi^n}{\sqrt{n!}} |n\rangle = \phi\,|\psi\rangle$

This state is called a <u>coherent</u> state.

We will call this state $|\phi\rangle$.

$$|\phi\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{\sqrt{n!}} |n\rangle = \sum_{n=0}^{\infty} \frac{\phi^n (\hat{a}^+)^n}{n!} |0\rangle$$

$$= \exp[\phi\, \hat{a}^+]\, |0\rangle$$

similarly $\langle \phi| = \langle 0|\, \exp[\phi^* \hat{a}]$

and $\langle \phi|\hat{a}^+ = \langle \phi|\,\phi^*$

so $$\hat{a}^+|\phi\rangle = \sum_{n=0}^{\infty} \hat{a}^+ \frac{(\hat{a}^+)^n \phi^n}{n!} |0\rangle$$

$$= \sum_{n=1}^{\infty} \frac{(\hat{a}^+)^n \phi^{n-1}}{(n-1)!} |0\rangle$$

$$= \frac{d}{d\phi} \sum_{n=1}^{\infty} \frac{(\hat{a}^+)^n \phi^n}{n!} |0\rangle$$

$$= \frac{d}{d\phi} \sum_{n=0}^{\infty} \frac{(\hat{a}^+ \phi)^n}{n!} |0\rangle$$

$$= \frac{d}{d\phi} |\phi\rangle$$

similarly $\langle \phi|\hat{a} = \langle \phi| \overleftarrow{\frac{\partial}{\partial \phi^*}}$

overlap and normalization

$$\langle \phi | \phi' \rangle = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{(\phi^*)^n}{\sqrt{n!}} \frac{(\phi')^{n'}}{\sqrt{n'!}} \underbrace{\langle n | n' \rangle}_{\delta_{nn'}}$$

$$= \sum_{n=0}^{\infty} \frac{(\phi^* \phi')^n}{n!} = \exp[\phi^* \phi']$$

$\Rightarrow$ in general coherent states are not orthogonal

to normalize multiply by $e^{-\frac{1}{2}|\phi|^2}$ to get $\langle \phi_{norm} | \phi_{norm} \rangle = 1$.

The coherent states are overcomplete with the following completeness relation

$$\int \frac{d\phi^* d\phi}{2\pi i} e^{-|\phi|^2} |\phi\rangle\langle\phi| = \mathbb{1}$$

check: $\dfrac{d\phi^* d\phi}{2\pi i} = \dfrac{d\,\mathrm{Re}\phi \;\; d\,\mathrm{Im}\phi}{\pi} = \dfrac{da\,db}{\pi}$ ✓

$\phi = a + ib \quad \phi^* = a - ib \qquad \dfrac{\partial\, \phi, \phi^*}{\partial\, a, b} = \begin{pmatrix} 1 & -i \\ i & +i \end{pmatrix}$ Jac det $= +2i$

So $\displaystyle \int da \int db \frac{1}{\pi} e^{-a^2-b^2} \sum_{nn'=0}^{\infty} \frac{\phi^n}{\sqrt{n!}} |n\rangle\langle n'| \frac{(\phi^*)^{n'}}{\sqrt{n'!}}$

But $\displaystyle \int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} db \frac{1}{\pi} e^{-a^2-b^2} (a+ib)^n (a-ib)^{n'} = I_{nn'}$

$= \delta_{nn'}\, n!$

The identity is not obvious, so let's prove it

suppose $n \geq n'$. convert from $d\alpha\, d\beta$ to $d\rho\, d\theta$
cartesian polar

$$I_{nn'} = \int_0^{\infty} d\rho \int_0^{2\pi} \rho\, d\theta\, \frac{1}{\pi} e^{-\rho^2} (\rho^2)^{n'} \rho^{n-n'} e^{i(n-n')\theta}$$

$\int_0^{2\pi} e^{i(n-n')\theta} d\theta$

This integral vanishes when integrated over θ unless $n-n'=0$

where it is 2π when integrated over θ

so

$$I_{nn'} = \delta_{nn'} \int_0^{\infty} d\rho\, \frac{2\pi}{\pi} e^{-\rho^2} \rho (\rho^2)^n. \qquad \rho^2 = u \quad 2\rho\, d\rho = du$$

$$= \delta_{nn'} \int_0^{\infty} du\, e^{-u} u^n = \delta_{nn'} \frac{d^n}{d\alpha^n} (-1)^n \int_0^{\infty} e^{-\alpha u} du$$

$$= \delta_{nn'} \frac{d^n}{d\alpha^n} (-1)^n \frac{1}{\alpha}$$

$$= \delta_{nn'}\, n!$$

as claimed.

$$\text{so} \quad \int \frac{d\phi^* d\phi}{2\pi i} e^{-|\phi|^2} |\phi\rangle\langle\phi| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \underline{\mathbb{1}} \quad \checkmark$$

let $\hat{O}$ be an operator. Then

$$\mathrm{Tr}\, \hat{O} = \sum_n \langle n|\hat{O}|n\rangle = \int \frac{d\phi^* d\phi}{2\pi i} e^{-|\phi|^2} \langle\phi|\hat{O}|\phi\rangle$$

Consider a wave function $|\psi\rangle$. We can write

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle = \int \frac{d\phi^* d\phi}{2\pi i}\, e^{-|\phi|^2} |\phi\rangle\langle\phi|\psi\rangle$$

call $\langle\phi|\psi\rangle = \psi(\phi^*) =$ number. Then

$$|\psi\rangle = \int \frac{d\phi^* d\phi}{2\pi i}\, e^{-|\phi|^2}\, \psi(\phi^*)\, |\phi\rangle$$

is the expansion of $|\psi\rangle$ in the coherent state basis.

similarly $\quad \langle\phi|\hat{a}|f\rangle = \dfrac{\partial}{\partial\phi^*} f(\phi^*)$

proof: $\langle\phi|\hat{a}|f\rangle = \displaystyle\int \frac{d\phi'^* d\phi'}{2\pi i}\, e^{-|\phi'|^2} f(\phi'^*)\, \langle\phi|\hat{a}|\phi'\rangle$

$$= \int \frac{d\phi'^* d\phi'}{2\pi i}\, e^{-|\phi'|^2} f(\phi'^*)\, \phi'\, e^{\phi^*\phi'}$$

$$= \frac{d}{d\phi^*} \int \frac{d\phi'^* d\phi'}{2\pi i}\, e^{-|\phi'|^2} \langle\phi|\phi'\rangle\langle\phi'|f\rangle$$

$$= \frac{d}{d\phi^*}\langle\phi|f\rangle = \frac{d}{d\phi^*} f(\phi^*) \quad \checkmark$$

and $\quad \langle\phi|\hat{a}^\dagger|f\rangle = \displaystyle\int \frac{d\phi'^* d\phi'}{2\pi i}\, e^{-|\phi'|^2} f(\phi'^*)\, \langle\phi|\hat{a}^\dagger|\phi'\rangle$

= ~~integrate by parts~~ $\qquad \langle\phi|\hat{a}^\dagger = \phi^*\langle\phi|$

$$= \phi^* f(\phi^*)$$

So we have symbolically

$$\hat{a} \leftrightarrow \frac{d}{d\phi^*} \qquad \hat{a}^+ \leftrightarrow \phi^*$$

Which is consistent with

$$[\hat{a}, \hat{a}^+]_- = 1 \quad \Rightarrow \quad \left[\frac{d}{d\phi^*}, \phi^*\right]_- = 1$$

So $\frac{d}{d\phi^*}$ and ϕ^* are like $\hat{x}$ and $\hat{p}$ up to a factor of i !

If we have a Hamiltonian expressed in terms of $\hat{a}^+$ and $\hat{a}$ then (all $\hat{a}^+$ terms to L of $\hat{a}$ terms called normal-ordered form)

$$H(\hat{a}^+, \hat{a})|\psi\rangle = E|\psi\rangle$$

becomes

$$\langle\phi| H(\hat{a}^+, \hat{a})|\psi\rangle = E\,\psi(\phi^*)$$

$$\langle\phi| H\left(\phi^*, \frac{\partial}{\partial\phi^*}\right)|\psi\rangle = E\,\psi(\phi^*)$$

$$\boxed{H\left(\phi^*, \frac{\partial}{\partial\phi^*}\right)\psi(\phi^*) = E\,\psi(\phi^*)}$$

Is the new form of the Schroedinger equation.

If we instead use completeness, then

$$\int \frac{d\phi'^* d\phi'}{2\pi i} e^{-|\phi'|^2} \langle\phi| H\left(\phi^*, \frac{\partial}{\partial\phi^*}\right)|\phi'\rangle\langle\phi'|\psi\rangle$$

$$= \int \frac{d\phi'^* d\phi'}{2\pi i} e^{-|\phi'|^2} H\left(\phi^*, \frac{\partial}{\partial\phi^*}\right) e^{\phi^*\phi'} \psi(\phi'^*)$$

But all $\frac{\partial}{\partial\phi^*}$ terms are replaced by ϕ' $\left(\frac{\partial}{\partial\phi^*} e^{\phi^*\phi'} = \phi' e^{\phi^*\phi'}\right)$

So we get

$$\int \frac{d\phi'^* \, d\phi'}{2\pi i} e^{-|\phi'|^2} H(\phi^*, \phi') e^{\phi^* \phi'} \psi(\phi'^*)$$

so one doesn't need to evaluate derivatives at all.

Similarly, if $\hat{O}(\hat{a}^\dagger, \hat{a})$ is normal ordered, then

$$\langle \phi | \hat{O}(\hat{a}^\dagger, \hat{a}) | \phi' \rangle = O(\phi^*, \phi') e^{\phi^* \phi'}$$

$$\langle \phi | \hat{a}^\dagger \hat{a} | \phi \rangle = \phi^* \phi$$

The variance is $\langle \phi | (\hat{a}^\dagger \hat{a})^2 | \phi \rangle - \left(\langle \phi | \hat{a}^\dagger \hat{a} | \phi \rangle \right)^2$

$$= \langle \phi | \hat{a}^{\dagger 2} \hat{a}^2 | \phi \rangle + \langle \phi | \hat{a}^\dagger \hat{a} | \phi \rangle - (\phi^* \phi)^2$$

$$= (\phi^* \phi)^2 + \phi^* \phi - (\phi^* \phi)^2$$

$$= \phi^* \phi$$

Please look at the book chapter, which generalizes all results to coherent states for multiple bosons $\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots$

The formulas are very similar, but I will not rewrite them.