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The 1925 revolution of matrix mechanics and how to celebrate it in modern quantum mechanics classes[a)]

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In 1925, Heisenberg, Born, and Jordan developed matrix mechanics as a strategy to solve quantum-mechanical problems. While finite-sized matrix formulations are commonly taught in quantum instruction, following the logic and detailed steps of the original matrix mechanics has become a lost art. In preparation for the 100th anniversary of the discovery of quantum mechanics, we present a modernized discussion of how matrix mechanics is formulated, how it is used to solve quantum-mechanical problems, and how it can be employed as the starting point for a postulate-based formulation of quantum-mechanics instruction. We focus on the harmonic oscillator to describe how quantum mechanics advanced from the Bohr–Sommerfeld quantization condition, to matrix mechanics, to the current abstract ladder-operator approach. We also describe a number of different activities that can be included in the quantum mechanics classroom to celebrate this centennial.
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I. INTRODUCTION

Around the turn of the 20th century, atomic physics was in disarray. There were theories from Lord Raleigh based on vortices, orbital theories that involved huge numbers of electrons, even for the smallest atoms, plum-pudding models, and so on. Even spectroscopy had many mysteries including the discovery by Pickering of apparent half-odd integer quantum numbers for hydrogen in some blue stars. What was needed was more experimental input. Soon after the Rutherford experiment, Bohr emerged with his theory, and although flawed, it ushered in a great period of enlightenment over the ensuing decades for atomic theory.

Because classical mechanics, electromagnetism, and statistical mechanics were the main areas of physics around 1900, practitioners were experts in many of these fields. The idea that emerged at that time was that quantum mechanics was a *restriction* on the laws of motion for classical mechanics. Ultimately, the notion of Bohr and Sommerfeld was that adiabatic invariants are what is quantized. We will see how this Bohr–Sommerfeld quantization rule (quantum condition) gets quantum mechanics right for some systems in a "broad-brush" fashion, but needs to be corrected in some way (contemporaries called it *sharpening the quantum condition*). In addition, there was heavy use of Bohr's correspondence principle to ensure the smooth transition from quantum mechanics to classical mechanics.

In this work, we do not take a historical approach to the problem. Instead, we focus on the logic in a modern setting. This is not to say that history is unimportant. It is. What we want to emphasize is how to use the breakthrough ideas from 1925 in quantum instruction today. We will show that this is not only feasible, but it also can make a postulate-based approach to quantum mechanics more familiar and easier to understand for the student.

There are many references that discuss the history of matrix mechanics. We mention some of the notable ones here, to provide readers with resources to learn more about its rich history. There are two resources that do a fantastic job of clearly explaining the philosophy that led to matrix mechanics and the details of how the theory works. They are the first volume of Tomonaga's quantum mechanics book[1] and the written summary of Max Born's lectures[2] in MIT toward the end of 1925. In more recent textbooks, we wish to highlight the discussions in Longair,[3] Weinberg,[4] and Golub and Lamoreaux,[5] as being particularly thorough. The article by Fedak and Prentiss[6] also provides a wonderful historical and scientific account of how matrix mechanics emerged. A more complete historical account can be found in the study by Duncan and Janssen.[7] These materials provide a rather thorough reading list for those who wish to examine the topic further.

We envision the work described here as being most appropriate to use with graduate students, who already have seen quantum mechanics at least once before. Having some background knowledge makes it easier to put matrix mechanics into the broader context of quantum mechanics.

II. BOHR–SOMMERFELD QUANTIZATION

As we mentioned in the introduction, the Bohr–Sommerfeld quantization rule can be viewed as quantizing adiabatic invariants from classical mechanics. Most of us today are not familiar with adiabatic invariants, so we start with a short review of the idea by following Crawford's wonderful treatment.[8] We consider a simple harmonic oscillator, with mass M and frequency ω, which has a frequency that changes slowly in time ($d\omega/dt \neq 0$) due to a time varying spring constant. The energy is the sum of the kinetic and potential energies and is given by $E = (1/2)M\dot{x}^2 + (1/2)M\omega^2x^2$. It is not conserved in time, because the frequency is changing. The time derivative of the energy is given by

$$\frac{dE}{dt} = M\ddot{x}\dot{x} + M\omega^2 x\dot{x} + M\omega x^2\frac{d\omega}{dt}. \tag{1}$$

What we mean by the frequency changing slowly is that we can approximate $x(t) \approx x_0 \cos \omega t$, with a constant ω over one period. Then $\ddot{x} = -\omega^2 x$ during that periodic motion.

Substituting this result into Eq. (1) shows that the time derivative of the energy becomes

$$\frac{dE}{dt} = \frac{2PE}{\omega}\frac{d\omega}{dt}. \tag{2}$$

Here, $PE = \frac{1}{2}M\omega^2x^2$ is the *instantaneous* potential energy. Because the change in the frequency is so slow, we average the derivative above over a period to find

$$\frac{dE}{dt} = \frac{2\langle PE\rangle}{\omega}\frac{d\omega}{dt}, \tag{3}$$

with the angle brackets indicating an average over the period of the classical motion. The next step requires us to use the virial theorem. We start by differentiating the virial $Mx\dot{x}$ to find

$$\begin{aligned}\frac{d}{dt}(Mx\dot{x}) &= M\dot{x}^2 + Mx\ddot{x} = M\dot{x}^2 - M\omega^2x^2 \\ &= 2KE - 2PE,\end{aligned} \tag{4}$$

where, again, we assume the acceleration is unchanged by the change in the frequency with time over one period. Next, we integrate this result over one period, assuming the frequency is constant over the period. Because the motion is periodic, the position $x(t)$ and the velocity $\dot{x}(t)$ return to their initial values after one period, and the integral vanishes. This tells us that $\langle KE\rangle = \langle PE\rangle$ and then $E = \langle KE\rangle + \langle PE\rangle = 2\langle PE\rangle = 2\langle KE\rangle$. Substituting into Eq. (2) gives us

$$\frac{dE}{E} = \frac{d\omega}{\omega}, \quad \text{or} \quad \frac{E}{\omega} = C. \tag{5}$$

In words, even though the frequency is changing in time, the ratio of the total energy to the frequency remains constant, if the change in the frequency with time is slow. This approximate conservation law is called an *adiabatic invariant*,[9] and it played a large role in the early development of quantum mechanics. The restriction placed on classical mechanics for the quantum realm was to quantize the adiabatic invariant as being equal to $\hbar n$, where n is a nonnegative integer. Doing this gives us the energy of a quantum harmonic oscillator satisfying $E = \hbar\omega n$, which is incorrect, because it misses an additional quantum correction equal to $\hbar\omega/2$. Nevertheless, this was the prediction of Bohr–Sommerfeld quantization.[10,11] To see the relationship with the standard Bohr–Sommerfeld quantization, we consider the action integrated over a period $\oint pdq = \int_0^T M\dot{x}^2dt = 2\langle KE\rangle T$, where T is the period of the oscillation. Using $T = 2\pi/\omega$, we immediately find that $\oint pdq = 2\pi E/\omega = hn$, which is the standard Bohr–Sommerfeld quantization rule.

Note that this discussion required the motion to be periodic in order to apply the Bohr–Sommerfeld quantization rule, or to apply the quantization rule by using an adiabatic invariant. This focus on periodic orbits also motivated the use of a Fourier series to describe the motion in these orbits, because periodic motion can always be decomposed into the fundamental frequency $2\pi/T$ and its harmonics. Where this quantization strategy suffered was cases where the orbits were not periodic, and also when one had multiple adiabatic invariants to quantize. It further could not be used to solve all quantum problems.

The mindset of these early quantum researchers was that quantum mechanics arose as a restriction to the classical motion. The Bohr–Sommerfeld quantization condition works well for simple one-dimensional motion, but becomes more challenging for more complicated systems, even though it had the remarkable success of properly predicting the relativistic energy levels of hydrogen, which was only verified in quantum mechanics after the Dirac equation was discovered and solved. The whole procedure had many flaws to it, because it worked for some systems, but not for others—however, when it worked, it worked beautifully. It clearly needed to be revamped in some way to be able to describe the systems that it failed on. This was first solved with the advent of matrix mechanics in 1925.

Before moving to Sec. III, we make a comment about the symbol M, which is overloaded in this paper. The capital letter M is used to denote the mass, a bold capital letter $\mathbf{M}$ to denote a matrix, and the lower case letter m is used as a label for the index of a matrix.

III. MATRIX MECHANICS

The goal we have with this work is not to provide a historical account of how matrix mechanics works. Indeed, there are many such accounts, including three recent ones.[3,5,7] Instead, our focus is on translating the logic and beauty of matrix mechanics into a more modern setting, where we can use it in modern classroom discussion. As we describe below, one can even start a discussion about how a classical system is quantized by following a different set of postulates than the standard ones used in most textbooks. We find this approach to be more appealing because it provides a more physical understanding for the origin of quantum mechanics. The prerequisite is that one needs to have a firm grasp of Hamiltonian mechanics to understand the details of the derivations (although it is sufficient to just have a firm grasp of the equations of motion, which can be motivated without requiring a full course on intermediate classical mechanics). In this light, we focus our effort on the original two articles by Heisenberg[12] and Born and Jordan.[13] However, we translate the older ideas (quantum mechanics is a restriction of classical mechanics) to a more modern perspective (everything is quantum and classical mechanics emerges via the correspondence principle in specific limits).

In classical mechanics, periodic motion is described in terms of the fundamental frequency $\omega = 2\pi/T$ and its harmonics $n\omega$, because all functions periodic in T can be expanded by such a Fourier series. If this held in quantum mechanics, then we would expect the transitions in atoms (which are given by energy differences) to produce spectra given by multiples of some fundamental frequency. However, spectra are not organized in this way. The Rydberg–Ritz combination principle phenomenologically showed that the spectroscopic lines appeared at energy differences between two energy levels.[14] Such a result is incompatible with a Fourier series expansion and points to the need for another approach that automatically enforces the combination principle. We will next see how Heisenberg reformulated kinematical ideas to naturally incorporate the Rydberg–Ritz combination principle.

While Heisenberg was not aware of matrices (he called them tableau in his first article), a matrix structure naturally allows you to work with energy differences. Here is how. Heisenberg introduced dynamical matrices that depended on

time. By having the (m,n) matrix element depend on time as $\exp(i\omega_{mn}t)$, with $\omega_{mn} = (E_m - E_n)/\hbar$, we immediately achieve the goal that the matrix elements oscillate as the energy differences (in a modern language, we construct the matrix elements of the operator via $\langle m(t)|\hat{O}|n(t)\rangle$, with $|n(t)\rangle$ an energy eigenstate that oscillates according to the time-dependent Schrödinger equation via $|n(t)\rangle = \exp(-iE_nt/\hbar)|n(0)\rangle$ and with the state $|n(0)\rangle$ either determined via operator methods, or from the wavefunction solution to the time-independent Schrödinger equation—but Heisenberg knew nothing about energy eigenstates and only required this specific time dependence of the matrix elements). The key aspect of this choice that makes the approach viable is if we look at the matrix that represents a product of two operators, we form it by matrix multiplication of the two dynamical matrices. Then, one can immediately see that the (m,n) matrix elements of the product have the same time dependence as each of the individual matrices, so the time dependence is preserved under multiplication of the matrices (this follows because $\omega_{ml} + \omega_{ln} = \omega_{mn}$). It is remarkable that Heisenberg rediscovered matrices and matrix multiplications by trying to make a theory consistent with the Rydberg–Ritz combination principle! (Note that Heisenberg recognized that the spectra corresponded to the case where $E_m > E_n$, so he extended the matrix definition to $E_m < E_n$ by requiring the matrices to be Hermitian.) Another consequence of Heisenberg's assumption is that a time-independent Hamiltonian corresponds to a diagonal matrix with the energies E_n along the diagonal, as any other matrix form would have time dependence, and the energies must be governed by the Hamiltonian. Having made the decision to introduce matrices as the dynamical variables for quantum mechanics, one next needs to decide how to work with them. One issue that arises is that matrix multiplication is not commutative. What is the implication of this for the quantum theory?

In 1925, the Born–Sommerfeld quantization rules were believed to be correct, so Heisenberg used them to shed light on the canonical commutation relation between these position and momentum matrices. However, it was awkward to formulate matrices into an integral that leads to the action. Instead, Heisenberg decided to differentiate the integrated Bohr–Sommerfeld quantization condition to find the new quantum condition. Since the quantization condition depends on integers, he needed to generalize the idea of differentiating integer-valued objects. The hint he had for this was the Born correspondence principle, worked out in 1924, which replaces derivatives of integer-valued objects by finite differences.[15] Using this approach, he was able to establish that the commutation relation between position $\mathbf{q}$ and momentum $\mathbf{p}$ matrices had diagonal matrix elements equal to $i\hbar$.

"Differentiating" the Bohr–Sommerfeld quantization rule makes the commutator $[\mathbf{q},\mathbf{p}]$, here thought of in terms of infinite-dimensional matrices $\mathbf{q}$ and $\mathbf{p}$, of primary importance (especially because it was derived from the venerated Bohr–Sommerfeld quantization rules). Heisenberg showed that the new (or as he said sharpened) quantum condition required all diagonal matrix elements of $[\mathbf{q},\mathbf{p}]$ to be equal to $i\hbar$.[12] Soon thereafter, Born and Jordan showed that the matrix is diagonal.[13]

However, one does not need to base the reasoning for the quantum theory on the flawed Bohr–Sommerfeld quantization condition. Instead, we proceed via two modernized postulates: (1) the dynamical quantities of position and momentum are described by Hermitian matrices, with a harmonic time dependence given by the energy differences between energy levels (Heisenberg's breakthrough postulate) and (2) the classical Hamiltonian equations of motion continue to hold for the quantum objects as matrices (Ehrenfest's theorem). The first postulate enforces the Rydberg–Ritz combination principle from spectroscopy, while the second can be viewed as being the correspondence principle, assuming we can find a way to ultimately link these matrices to classical calculations.

One can now determine the canonical commutation relation by simply using postulate (2), in that the classical equations of motion hold for these quantum matrices. The first step is to show that the commutator is a constant in time (as originally shown by Born and Jordan). We find

$$\frac{d}{dt}[\mathbf{q},\mathbf{p}] = [\dot{\mathbf{q}},\mathbf{p}] + [\mathbf{q},\dot{\mathbf{p}}] = \frac{1}{M}[\mathbf{p},\mathbf{p}] + \left[\mathbf{q}, -\frac{d}{d\mathbf{q}}V(\mathbf{q})\right] = 0, \tag{6}$$

because momentum commutes with itself and position commutes with functions of position. (One might ask, How do we take the derivative in the second term? The easiest way to do this is to think of taking the derivative as we would in classical mechanics and substituting the matrices into the expression after the derivatives have been taken.) Next, because of the time-dependent structure of the matrices, where the matrix elements vary harmonically as $e^{i\omega_{mn}t}$, we have that the time derivative of the matrix element is given by $\dot{\mathbf{M}}_{mn} = i\omega_{mn}\mathbf{M}_{\mathrm{mn}}$. Now, because the Hamiltonian matrix is diagonal and equal to $E_m\delta_{mn}$, the *Heisenberg equation of motion* for generic matrices is given by

$$\frac{d}{dt}\mathbf{M}_{mn} = i\omega_{mn}\mathbf{M}_{mn} = \frac{i(E_m - E_n)}{\hbar}\mathbf{M}_{mn} = \frac{i}{\hbar}[\mathbf{H},\mathbf{M}]_{mn}. \tag{7}$$

We next apply the Heisenberg equation of motion to the commutator of position with momentum, which says the time derivative of the commutator is equal to $(i/\hbar)[\mathbf{H},[\mathbf{q},\mathbf{p}]]$, or because we are working in the basis where the Hamiltonian matrix is diagonal, we find the mn matrix element to be

$$0 = \frac{d}{dt}[\mathbf{q},\mathbf{p}]_{mn} = \frac{i}{\hbar}\left[\mathbf{H},[\mathbf{q},\mathbf{p}]\right]_{mn} = \frac{i(E_m - E_n)}{\hbar}[\mathbf{q},\mathbf{p}]_{mn}. \tag{8}$$

Hence, when $m \neq n$, we immediately learn that the commutator matrix element vanishes, so it is a diagonal matrix (at least in the basis where $\mathbf{H}$ is diagonal; we will later see it remains diagonal in all bases). Note that there is a tacit assumption here that the spectrum of the problem is nondegenerate. We know this is true for one-dimensional problems, in general, so one should focus this discussion on one-dimensional problems (on the infinite one-dimensional domain). For the diagonal matrix elements, Heisenberg, Born, and Jordan used the differentiation of the Bohr–Sommerfeld quantization rule to infer that the diagonal elements are all equal to $i\hbar$. However, that requires acceptance of Bohr–Sommerfeld quantization as the true quantum condition. Taking a more modern viewpoint, we can find the

diagonal elements by modifying an argument Herbert Green used to determine them.[16] We first examine the relationship between momentum and position via

$$\mathbf{p} = M\dot{\mathbf{q}} = i\frac{M}{\hbar}[\mathbf{H}, \mathbf{q}] = i\frac{M}{\hbar}\left[\frac{\mathbf{p}^2}{2M} + V(\mathbf{q}), \mathbf{q}\right] = -\frac{i}{2\hbar}(\mathbf{p}[\mathbf{q}, \mathbf{p}] + [\mathbf{q}, \mathbf{p}]\mathbf{p}) \tag{9}$$

after using the Leibniz product rule for the commutator (valid for operators and matrices). Now, we recall that $[\mathbf{q}, \mathbf{p}]$ is a diagonal matrix, and we let $\mathbf{c}_m$ denote its mth diagonal matrix element. Looking at the matrix elements in Eq. (9), we find

$$i\hbar\mathbf{p}_{mn} = \frac{1}{2}(\mathbf{c}_m + \mathbf{c}_n)\mathbf{p}_{mn} \tag{10}$$

valid for all m and n. Note that one cannot conclude that the canonical commutation relation is proportional to the identity matrix at this point, as there are many possible solutions to this equation. For example, one choice is indeed $c_m = i\hbar$, so the commutator is proportional to the identity matrix. But other choices may be possible too. If $\mathbf{p}_{mn} \neq 0$ only when m is odd and n is even and *vice versa* (as happens in the harmonic oscillator, for example), one could choose $c_{2m} = 2i\hbar$ and $c_{2m+1} = 0$. Nevertheless, it is only the solution proportional to the identity matrix that works, because the result must hold for all possible Hamiltonians. As we change from one Hamiltonian to another, the basis changes for which the Hamiltonian is diagonal. The canonical commutation relation must still be a diagonal matrix in this new basis (because our proof that it is a diagonal matrix did not depend on the form of the potential energy). The only matrix that maintains the same diagonal form regardless of the basis change is the identity matrix. So, the only possible choice for the solution is that the commutator is proportional to the identity because the commutator is a diagonal matrix for all possible Hamiltonians.

You might ask, is this an independent derivation of the canonical commutation relation? Unfortunately, it is not without using postulate (2)—the Ehrenfest theorem—that the expectation values of the classical equations of motion hold for the quantum system, when position and momentum are expressed in terms of dynamical matrices. These Ehrenfest relations are usually derived by using the canonical commutation relation within a standard quantum treatment of this problem. Here, we instead postulate that the classical equations of motion hold for these dynamical matrices. This is how there is an underlying postulate hidden within this "derivation." However, this postulate might be more physical and meaningful than directly postulating the canonical commutation relation, as is commonly done in quantum instruction. We believe it is interesting to share this alternative point of view with students, especially in the anniversary year of the discovery of quantum mechanics. We display the correspondence graphically in Fig. 1. Postulate-based approaches are common in instruction and appear in many textbooks but are often scattered through the text, rather than emphasized. One exception is Gillespie's book, which is entirely about postulates.[17] Interestingly, in most postulate formulations, the canonical commutation relation is not listed as a separate postulate, although it appears in a postulated form, either from postulating the representation of momentum as an operator or through Dirac's quantization rules with respect to the Poisson bracket.

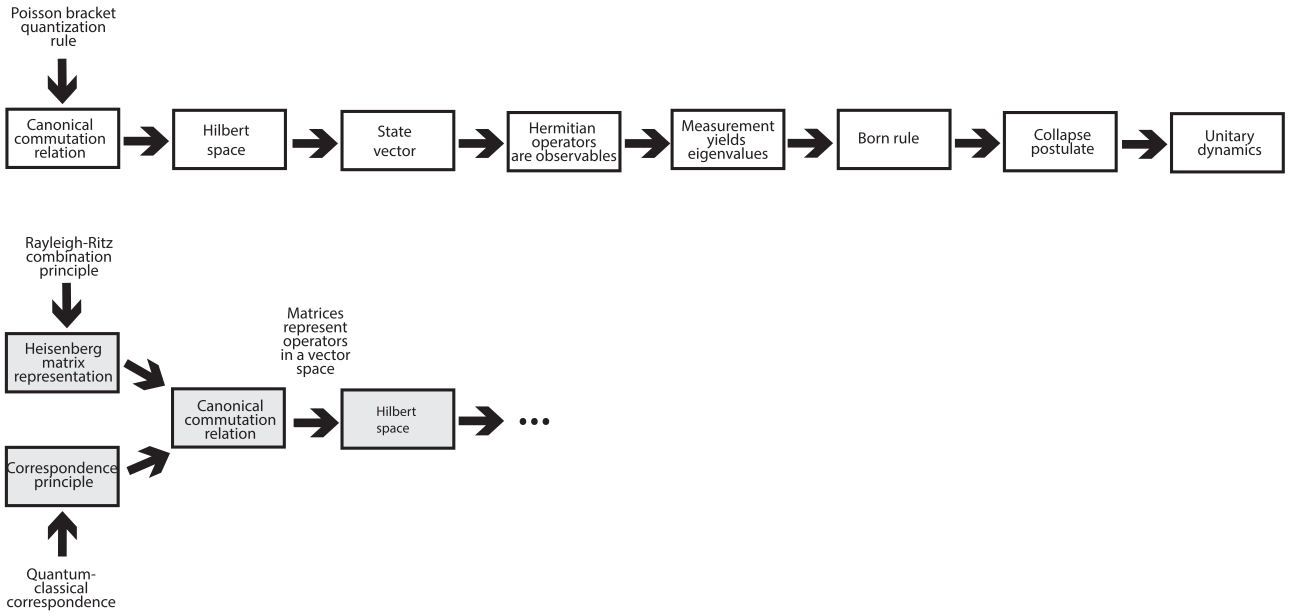

Fig. 1. Comparison of conventional (top) and matrix-mechanics (bottom) based formalism for quantum mechanics. In the conventional approach (top row), the canonical commutation relation is either postulated or proposed as a quantization rule from Poisson brackets. Then we postulate to work in a Hilbert space, with state vectors describing the physical system and Hermitian operators corresponding to observables. Next the measurement hypothesis describes how the result of any shot of an experiment is given by an eigenvalue of the corresponding operator, with the Born rule providing the probability that it will occur. The wavefunction is then collapsed onto the measured state. Finally, dynamics are unitary, governed by the time-dependent Schrödinger equation. These postulates have virtually no physical motivation. Our proposed revision (bottom row) starts from the Rayleigh–Ritz correspondence principle, which motivates describing the system with matrices, and the correspondence principle, which can be used to derive the canonical commutation relation by employing the Hamilton equations of motions to the matrices. Since matrices represent operators in some abstract vector space, we then pivot to working in that abstract space, following the standard progression thereafter.

One of the challenges of the matrix mechanics approach is that because we work in a basis where the Hamiltonian is diagonal, we need to use the other information we have to concretely construct the Hamiltonian matrix (how to go beyond this is touched upon in the "Drei-Mann Arbeit" paper[18]). This can only be done for the harmonic oscillator and angular momentum eigenstates. The angular momentum case is discussed in many textbooks as the operator method to find states of definite angular momentum. We focus here on the harmonic oscillator solution because it is done differently from how we see it in modern textbooks.

The harmonic oscillator Hamiltonian satisfies

$$\mathbf{H} = \frac{1}{2M}\mathbf{p}^2 + \frac{M\omega^2}{2}\mathbf{q}^2. \tag{11}$$

The equation of motion becomes $M\ddot{\mathbf{q}} = -M\omega^2\mathbf{q}$, or $(\omega_{mn}^2 - \omega^2)\mathbf{q}_{mn} = 0$. Hence, whenever $\mathbf{q}_{mn} \neq 0$, we must have $\omega_{mn} = \pm\omega$. Use "0" to label the lowest energy level and "1" for the state that it is coupled to. Then $\omega_{01} = -\omega \neq 0$, so $\mathbf{q}_{0n} \neq 0$ only for $n = 1$ (recall the energies are nondegenerate). The next step we have is to rewrite the canonical commutation relation in terms of the matrices, using $\mathbf{p} = M\dot{\mathbf{q}}$. This gives

$$[\mathbf{q}, \mathbf{p}]_{mn} = iM\sum_l (\omega_{ln} - \omega_{ml})\mathbf{q}_{ml}\mathbf{q}_{ln} = i\hbar\delta_{mn}. \tag{12}$$

Choose $m = n$ to find that

$$iM\sum_l (\omega_{lm} - \omega_{ml})|\mathbf{q}_{ml}|^2 = i\hbar,$$

or

$$\sum_l \omega_{ml}|\mathbf{q}_{ml}|^2 = -\frac{\hbar}{2M}. \tag{13}$$

From this, we learn that $|\mathbf{q}_{01}|^2 = \hbar/2M\omega$ because only the 01 matrix element is nonzero. This means for the 0 column and row, all elements of $\mathbf{q}$ vanish except $\mathbf{q}_{01} = e^{-i\omega t}\sqrt{\hbar/2M\omega}$ and then Hermiticity tells us that $\mathbf{q}_{10} = e^{i\omega t}\sqrt{\hbar/2M\omega}$. Similar result holds for $\mathbf{p}$, with $\mathbf{p}_{01} = -ie^{-i\omega t}\sqrt{\hbar M\omega/2}$ and $\mathbf{p}_{10} = ie^{i\omega t}\sqrt{\hbar M\omega/2}$. So, we can use these two results to now calculate the 00 matrix element of $\mathbf{H}$. We find

$$\mathbf{H}_{00} = \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} = \frac{1}{2}\hbar\omega. \tag{14}$$

This is the well-known result for the ground-state energy of the harmonic oscillator. Since all higher energies are incremented by $\hbar\omega$, we immediately find that $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$. We can fill in the remaining matrix elements for the position and momentum matrices, but we do not go through this exercise here.

In the modern language, we have

$$\mathbf{q}_{mn} = \langle m(t)|\hat{q}|n(t)\rangle \text{ and } \mathbf{p}_{mn} = \langle m(t)|\hat{p}|n(t)\rangle, \tag{15}$$

where $|n\rangle$ satisfies $\hat{H}|n\rangle = E_n|n\rangle$ and the time-dependent state is $|n(t)\rangle = \exp(-iE_nt/\hbar)|n\rangle$. From this, you can immediately see we capture the proper time dependence, the Hermiticity, and the other properties that Heisenberg postulated. Furthermore, we also know that the requirement that the Hamiltonian matrix be diagonal can be relaxed—we can compute the position and momentum matrices in any basis, and from them, we can compute the Hamiltonian matrix. Then, we simply have to diagonalize the Hamiltonian matrix to determine the energy eigenstates and eigenvalues. The challenge is that this is difficult to carry out for infinite dimensional matrices. Nevertheless, this was still viewed as a valid way of solving these problems. Tomonaga's text[1] describes this procedure in detail in the final section of Chapter 5. Alternatively, we can truncate the matrix (by removing the high-energy contributions) to make it finite and diagonalize it numerically. This is a common way to approximately solve the Schrödinger equation.

We want to reiterate that there is no wavefunction in this approach, or even quantum eigenstate, just dynamical matrices. It is amazing that one can solve any problem with such restrictions, and this is in many respects the beauty of matrix mechanics.

IV. LADDER OPERATORS

Many of you might be puzzling at this point—where are the ladder operators? They emerged a bit later. The first we found of them is in a 1928 paper by Jordan and Pauli, which focused on quantizing light.[19] There, the conventional raising and lowering operators are defined but are thought of as infinite-dimensional matrices. While the way they are used in solving the harmonic oscillator is familiar to all, we briefly discuss how this development enters into the matrix mechanics context, especially because these early approaches are different from how we currently do this.

The first extensive discussion of how to use ladder operators is by Born and Jordan in their 1930 textbook *Elementare Quantenmechanik*.[20] Their strategy is to convert the coupled first-order classical differential equations ($\dot{\mathbf{q}} = \mathbf{p}/M$ and $\dot{\mathbf{p}} = -M\omega^2\mathbf{q}$) into two decoupled first-order differential equations by defining $\mathbf{b} = C(\mathbf{p} - iM\omega\mathbf{q})$ and $\mathbf{b}^\dagger = C(\mathbf{p} + iM\omega\mathbf{q})$. These matrices satisfy the uncoupled differential equations $\dot{\mathbf{b}} = -i\omega\mathbf{b}$ and $\dot{\mathbf{b}}^\dagger = i\omega\mathbf{b}^\dagger$. They choose C so that $[\mathbf{b}, \mathbf{b}^\dagger] = 1$; that is, $C = 1/\sqrt{\hbar M\omega}$. Next, keeping track of the order of the matrices, they calculate the products of these operators to find that

$$\mathbf{H} = \hbar\omega\mathbf{b}\mathbf{b}^\dagger - \frac{\hbar\omega}{2} = \hbar\omega\mathbf{b}^\dagger\mathbf{b} + \frac{\hbar\omega}{2}. \tag{16}$$

To complete the solution of the problem, they explicitly compute the matrices for $\mathbf{b}$ and $\mathbf{b}^\dagger$, with each having only at most one nonzero matrix element in each row and column. Then, they can construct the Hamiltonian matrix $\mathbf{H}$ from the expressions in Eq. (16) and find that the energies of the harmonic oscillator are given by $\hbar\omega(n + 1/2)$.

There is another solution to this problem worked out in 1930 by Dirac in his textbook *Principles of Quantum Mechanics*,[21] which was released after Born and Jordan's book. Dirac forms raising and lowering matrices for a dimensionless Hamiltonian given by $\mathbf{P}^2 + \mathbf{Q}^2$, with $[\mathbf{Q}, \mathbf{P}] = i$. He defines $\mathbf{A} = (\mathbf{P} + i\mathbf{Q})(\mathbf{P} - i\mathbf{Q})$ and by examining $\mathbf{A}(\mathbf{P} + i\mathbf{Q}) = (\mathbf{P} + i\mathbf{Q})(\mathbf{A} + 2)$, a form of an intertwining relation, he performs expansions of this relation in terms of eigenstates of $\mathbf{A}$ and is able to reason that the eigenvalues of $\mathbf{A}$ are 0, 2, 4, … and hence of $\mathbf{P}^2 + \mathbf{Q}^2$ are 1, 3, 5, …. He does use the stepping up and down the spectra and the fact

that the lowest eigenvalue must be nonnegative in his discussion. He also computes explicit matrices for the ladder operators.

We only get to the modern treatment of this problem in the third edition of Dirac's book,[22] from 1947, and Schiff's 1949 textbook.[23] This is because both of these approaches required Schrödinger's results for the factorization method, which constructed the energy eigenstates in terms of raising operators acting on the ground state.[24] Their presentations had influence from Rojansky's 1938 book as well, which outlines about half of the abstract argument.[25]

V. CLASSROOM ACTIVITIES

We envision two ways of bringing this material into a single lecture in a quantum mechanics class. The first is to use this approach to describe quantum postulates in a more meaningful way. By linking the sharp lines in atomic spectra to the Rydberg–Ritz combination principle, one motivates using matrices, because they naturally have, and preserve, the harmonic time dependence that is consistent with this principle. One can argue that the electron in the atom oscillates at the frequency derived from the energy differences of the two energy levels, which then creates a photon from the oscillating electric field. Then, using the correspondence principle to bring in the Ehrenfest equations of motion and justifying using them due to the correspondence principle is also natural. Having a diagonal Hamiltonian matrix also follows for time-independent problems, because otherwise the Hamiltonian has explicit time dependence. This then allows one to *derive* the canonical commutation relation once we recognize that it is a diagonal matrix for all Hamiltonians. Then, to get to the rest of the quantum postulates, one shows how picking states in a Hilbert space (or Gelfand triple, if you want to be more rigorous), and abstract operators that map states to states, allows you to show how the matrices are constructed, and thereby complete the introduction of quantum ideas (one needs to also discuss measurement and the Born rule, of course). The second way, if done after instruction has started, is to motivate the matrices by working with time-dependent energy eigenstates, and then showing how they lead to the canonical commutation relation, so that if one uses the postulates as stated above, we could have introduced quantum mechanics this way. Once these initial ideas are covered, showing how to solve the harmonic oscillator this way is the other valuable topic to discuss.

The remainder of the suggestions are problems one can use to help students have contact with the early quantum mechanics ideas, especially with how they led to matrix mechanics.

One interesting problem from the early days of quantum mechanics is to analyze the Pickering–Fowler lines controversy. An astronomer, Pickering, found hydrogen-like lines with half-odd integer quantum numbers in a small class of hot blue stars. Fowler performed spectroscopy on earth and found they only occurred when he had helium as well as hydrogen in the experiment. A good problem is to have students use reduced mass in the analysis of actual measured spectra (just as Bohr did[26]) to see whether the Balmer series with half-odd integer quantum numbers, or singly ionized helium with integer quantum numbers fits the data better. This also brings in the idea that high-precision measurements often have impact beyond just the measurement itself. It is believed that this analysis is what convinced Einstein that the Bohr model must be correct.

Another good exercise is to have students derive the adiabatic invariant for the classical harmonic oscillator, and see why having that be quantized makes sense, as being the most "constant" object in classical mechanics. It has the benefit of requiring very little calculus to complete.

An interesting problem is to derive the energy eigenvalues for an anharmonic oscillator as Heisenberg did in his original paper. This would be given to students post instruction, especially if perturbation theory has already been covered. One could have the students do it the modern way with perturbation theory or the Heisenberg way with matrix mechanics.

Using the wavefunctions of energy eigenstates, one can ask students to solve for the position or momentum matrices for the harmonic oscillator, and for other solvable models, such as the particle in a box.

Finally, rather than just saying that raising and lowering operators are an ingenious trick that Dirac thought up, take the students through the rich history. It all starts with Heisenberg[12] and Born and Jordan,[13] who showed how to solve the harmonic oscillator using matrix mechanics. Then Schrödinger showed how to solve it using wavefunctions.[27] Next, Jordan and Pauli[19] introduce the ladder operators. Then Born and Jordan,[20] followed by Dirac,[21] included a matrix mechanics discussion in their original textbooks (although the methods each used are not the current standard methods). Even though these results can be interpreted as working with abstract operators, one can clearly see in both cases they were working with infinite-dimensional matrices (as they actually wrote down the matrices explicitly in both books). It was Rojansky's book[25] in 1938 that first discussed using this approach in terms of abstract operators, and finally, in 1940, Schrödinger[24] showed how to create the energy eigenstates directly from the operators when he developed the factorization method. This then started appearing in textbooks in a modern way after the Second World War starting with Dirac's third edition[22] and Schiff's textbook.[23]

VI. SUMMARY AND CONCLUSIONS

While matrix mechanics has tremendous beauty in its arguments, it is unknown how to generalize it to solve all problems, even numerically. To do so, one would need to essentially guess the form of the infinite-dimensional position and momentum matrices in the energy eigenstate basis in order to construct the general solution. Since this is essentially impossible, the number of problems that can be solved this way were limited to the harmonic oscillator and the angular momentum eigenvectors. When an alternative method emerged (wavefunctions and the Schrödinger equation), which had the added benefit of having a pictorial representation for the solutions, matrix mechanics was replaced by the Schrödinger approach.

A more interesting question is, Why didn't matrix mechanics morph into working directly with abstract operators, as we routinely do now for the harmonic oscillator and angular momentum? Indeed, Pauli's solution of hydrogen moved precisely in this direction, as his work should be viewed as working with abstract operators instead of matrices. Sadly, it seems that the issue was that extending the approach to other problems took another 15 years for Schrödinger to figure it out, so there was nothing available to

allow further development at that time. Once the new approach was found, the wave equation was so deeply entrenched it could not be moved.

We hope that readers agree that this material has an intrinsic beauty to it and a clear logic that begs for it to be introduced in a meaningful way into quantum instruction, even if it is not going to be used extensively. We believe that rethinking how to organize quantum postulates to allow the matrix mechanics story to be told is probably the best way to do so. Matrix mechanics remains important within quantum mechanics and is currently honored with our use of "matrix element" to describe $\langle n|\hat{O}|m\rangle$ and with the naming of the S-matrix in quantum field theory and the T-matrix in many-body physics, amongst other examples of the usage. With this work, we hope it can be honored even more!

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

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