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Natural relationship between classical orbits and quantum Hamiltonians for the Kepler problem

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Central force problems involve coupled linear differential equations of motion. By decoupling the differential equations, we show how to efficiently solve for the orbits of the Kepler problem without any complicated integrals. Introducing quantization provides a deep connection between the solution for these classical orbits and the quantum Hamiltonian of hydrogen for states that have definite total angular momentum. https://doi.org/10.1119/5.0220799

I. INTRODUCTION

The Hamilton equations of motion for central force problems are a set of coupled linear differential equations developed in every upper-division classical mechanics class. Even though these equations do not usually have constant coefficients, they can be decoupled using a strategy that was first described in the 1930 quantum textbook by Born and Jordan called *Elementare Quantenmechanik* for the simple harmonic oscillator.[1] The decoupling is closely related in form to the factorization method, which was introduced by Schrödinger in 1940, as an alternative way to solve for the energy levels of exactly solvable quantum problems.[2] The relationship we develop in this paper provides a deep connection between the mathematics of classical mechanical orbits and quantum mechanical energy eigenvalues, but it is not directly related to the Bohr–Sommerfeld quantization conditions. It is a different strategy that reveals an interesting relationship that is valuable for both students studying classical mechanics and quantum mechanics. We show this approach only for the Kepler problem of an inverse square force law. While it also works for the isotropic harmonic oscillator, the orbit equation analysis is significantly more complicated, so we do not discuss that case further here. A conventional classical analysis appears in Sec. 11.9 of a *Physics Libre* textbook.[3]

We start our treatment with the Hamilton equations of motion for central forces. The strategy then involves a decoupling of the linear differential equations, followed by the use of the conservation of angular momentum, which then allows for an immediate solution of the orbit in the Kepler problem.

Consider a classical particle of mass m moving in a central potential $V(r)$. The motion is restricted to a plane and can be described by the radial coordinate r and the angular coordinate θ. The kinetic energy becomes $T=\frac{1}{2}m\dot{r}^2+\frac{1}{2}\,mr^2\dot{\theta}^2 = p_r^2/2m+p_\theta^2/2mr^2$, with the radial momentum $p_r=m\dot{r}$ and the angular momentum $p_\theta=mr^2\dot{\theta}$. The Hamiltonian is $H=T+V$ (using the form $T=p_r^2/2m+p_\theta^2/2mr^2$), and the Hamilton equations of motion become

$$\dot{r}=\frac{\partial H}{\partial p_r}=\frac{p_r}{m} \quad \text{and} \quad \dot{p}_r=-\frac{\partial H}{\partial r}=mr\dot{\theta}^2-\frac{dV(r)}{dr} \tag{1}$$

for the radial degree of freedom and

$$\dot{\theta}=\frac{\partial H}{\partial p_\theta}=\frac{p_\theta}{mr^2} \quad \text{and} \quad \dot{p}_\theta=-\frac{\partial H}{\partial \theta}=-\frac{dV(r)}{d\theta}=0 \tag{2}$$

for the angular degree of freedom; details can be found in Goldstein.[4] The central force has no angular component, which is why the angular equations can be solved by $p_\theta=mr^2\dot{\theta}=L$, which says that the angular momentum perpendicular to the plane is a conserved quantity. The radial equations have contributions from the central force and the centrifugal barrier term. By using the conserved angular momentum, we find

$$\dot{p}_r=\frac{L^2}{mr^3}-\frac{dV(r)}{dr}. \tag{3}$$

II. THE KEPLER PROBLEM

For the Kepler problem, we have that $V(r)=-k/r$, and thus the set of equations

$$\dot{r}=\frac{p_r}{m} \quad \text{and} \quad \dot{p}_r=\frac{L^2}{mr^3}-\frac{k}{r^2}. \tag{4}$$

One of the strategies to solve coupled differential equations is to decouple them. This means that we wish to combine the equations in such a way that the time derivative depends on only the object being differentiated and some multiplicative factor. For example, the original equations have the derivative of the position proportional to the momentum and the derivative of momentum equal to a function of position. These are coupled equations. The equations are decoupled by the ansatz

$$A=p_r+\frac{\alpha}{r}+\beta, \tag{5}$$

with α having dimensions of angular momentum, and β having dimensions of linear momentum. We compute $\dot{A}$ and require it to be equal to $\lambda(r)A$ in order to decouple the differential equations. This yields, after using Eq. (4),

$$\dot{A} = \dot{p}_r - \frac{\alpha \dot{r}}{r^2} = \frac{L^2}{mr^3} - \frac{k}{r^2} - \frac{\alpha p_r}{mr^2}. \tag{6}$$

The coefficient of the radial momentum must be the factor λ. Once that has been determined, the other coefficients follow directly by matching corresponding terms. We find that

$$\lambda(r) = -\frac{\alpha}{mr^2}, \quad \alpha = -\frac{L^2}{\alpha}, \quad \text{and} \quad \beta = \frac{km}{\alpha}. \tag{7}$$

The middle equation is immediately solved by $\alpha = \pm iL$, and then $\lambda = \mp iL/mr^2$ and $\beta = \mp ikm/L$. We have two solutions, and they are complex conjugates of each other. We choose A to be defined with the $\alpha = iL$ case, and A^* with the $\alpha = -iL$ case. This gives

$$A = p_r + \frac{iL}{r} - \frac{ikm}{L}, \tag{8}$$

and A^* is the complex conjugate, corresponding to the other independent solution.

The differential equation has now become

$$\dot{A} = -\frac{iL}{mr^2}A \tag{9}$$

and its complex conjugate for A^*. This differential equation appears as if it cannot be directly solved, because we do not know $r(t)$. But, by using $L/mr^2 = \dot{\theta}$, we find that

$$\dot{A} = -i\dot{\theta}A, \tag{10}$$

which is immediately integrated to

$$A(t) = A_0 e^{-i\theta}, \tag{11}$$

with A_0 being the "initial value" of A at $\theta = 0$. One can think of the θ in the exponent as a function of t or as just θ itself for the equation of the orbit. The complex conjugate equation also holds.

Now to find the orbit, we eliminate p_r by taking the difference of $A(\theta)$ and $A^*(\theta)$. We find that

$$\begin{aligned} A(\theta) - A^*(\theta) &= -2i\,\mathrm{Re}(A_0)\sin\theta + 2i\,\mathrm{Im}(A_0)\cos\theta \\ &= \frac{2iL}{r} - \frac{2ikm}{L}, \end{aligned} \tag{12}$$

by evaluating the difference from the solutions to the differential equation and from the definitions of A and A^* in Eq. (8). Picking the initial value A_0 to correspond to a point on the orbit with zero radial momentum (and the distance of the closest approach to the center of the force), we have Re $(A_0) = 0$ and Im $(A_0) = (L/r_0) - (km/L)$, yielding the orbit equation

$$\frac{1}{r} = \frac{km}{L^2} + \left(\frac{1}{r_0} - \frac{km}{L^2}\right)\cos\theta. \tag{13}$$

This is an orbit equation for an ellipse with the origin at one of the foci. To see how it compares with the standard form in Eq. (3.55) of Goldstein's textbook, we simply need to re-express in terms of the energy.[4] The (constant) energy can be evaluated at the distance of closest approach, as given by

$$E = \frac{L^2}{2mr_0^2} - \frac{k}{r_0}, \tag{14}$$

because there is no radial momentum contribution there. Solving the quadratic equation for the distance of closest approach (which requires the plus sign for the specific root) yields

$$\frac{1}{r_0} = \frac{km}{L^2} + \frac{km}{L^2}\sqrt{1 + \frac{2EL^2}{mk^2}}, \tag{15}$$

which allows us to re-express the orbit equation as

$$\frac{1}{r(\theta)} = \frac{km}{L^2}\left(1 + \sqrt{1 + \frac{2EL^2}{mk^2}}\cos\theta\right). \tag{16}$$

Note that this derivation is substantially more elementary than the conventional approach, which requires a complicated derivation of $d\theta/dr$, followed by an integration to determine $\theta(r)$, and finally, an inversion to obtain $r(\theta)$ as seen in the study of Goldstein, for example Ref. 4. We discuss how to obtain the energy directly from A and A^* next.

III. TRANSITION TO QUANTUM MECHANICS

We start by examining the energy. We know the energy is a conserved quantity, and so is $A^*(\theta)A(\theta) = |A_0|^2$, because the complex conjugate exponentials cancel when multiplied together, so there should be a relationship between them. Using Eqs. (8) and (11), we find that

$$|A_0|^2 = \frac{L^2}{r_0^2} - \frac{2km}{r_0} + \frac{k^2m^2}{L^2} = p_r^2 + \frac{L^2}{r^2} - \frac{2km}{r} + \frac{k^2m^2}{L^2}, \tag{17}$$

which is equal to $2mE + k^2m^2/L^2$. By subtracting the constant k^2m^2/L^2 from both sides and dividing by $2m$, we use Eq. (14) to verify the standard conservation of energy for the Kepler problem. This shows that the decoupling used to find the orbits also allows us to derive the conservation of energy.

Finally, we show how this classical solution is related to the quantum problem for hydrogen. In quantum mechanics, the radial momentum is conjugate to the radial coordinate, so that $[\hat{r}, \hat{p}_r] = i\hbar$. Furthermore, the angular momentum is quantized and cannot take on continuous values. The surprising result is that we can use a quantized version of A to factorize the Hamiltonian for hydrogen. In the case of hydrogen, we take $k = e^2/4\pi\varepsilon_0 = \hbar^2/ma_0$, where e is the charge on an electron, and a_0 is the Bohr radius. Then, we construct the quantum operator to have the same form as the classical operator via

$$\hat{A}_l = \hat{p}_r + \frac{i\hbar(l+1)}{\hat{r}} - \frac{i\hbar}{a_0(l+1)} \tag{18}$$

after writing $L = \hbar(l+1)$. The symbol l will become an integer, but for now, think of it as a real dimensionless number, so there is no loss in generality. We next compute

$$\hat{A}_l^\dagger \hat{A}_l = \hat{p}_r^2 + i\hbar(l+1)\left[\hat{p}_r, \frac{1}{\hat{r}}\right] + \frac{\hbar^2(l+1)^2}{\hat{r}^2} - \frac{2\hbar^2}{a_0\hat{r}} + \frac{\hbar^2}{a_0^2(l+1)^2}$$
$$= \hat{p}_r^2 + \frac{\hbar^2 l(l+1)}{\hat{r}^2} - \frac{2\hbar^2}{a_0\hat{r}} + \frac{\hbar^2}{a_0^2(l+1)^2} \quad (19)$$

after evaluating the commutator, which satisfies

$$\left[\hat{p}_r, \frac{1}{\hat{r}}\right] = \frac{i\hbar}{\hat{r}^2}. \quad (20)$$

If we divide by $2m$ and subtract the constant from both sides, we find that

$$\hat{H}_l = \frac{1}{2m}\hat{A}_l^\dagger \hat{A}_l - \frac{\hbar^2}{2ma_0^2(l+1)^2}$$
$$= \frac{\hat{p}_r^2}{2m} + \frac{\hbar^2 l(l+1)}{2m\hat{r}^2} - \frac{\hbar^2}{ma_0\hat{r}}, \quad (21)$$

which is the standard radial quantum Hamiltonian when acting on states of definite total angular momentum l, with l being a non-negative integer. The constant term on the right-hand side of the top line is the energy of the ground state for the given l value because the operator part of the Hamiltonian is a positive semidefinite operator, whose minimal expectation value is 0 for states that satisfy $\hat{A}_l|\psi\rangle = 0$.

The Heisenberg equation of motion for $\hat{A}$ is found to be

$$\dot{\hat{A}} = \frac{i}{\hbar}\left[\hat{H}, \hat{A}\right] = \frac{i}{2m\hbar}\left[\hat{A}^\dagger, \hat{A}\right]\hat{A} = -\frac{i\hbar(l+1)}{m\hat{r}^2}\hat{A}, \quad (22)$$

which has the same form as the classical equation of motion in Eq. (9), with L identified with $\hbar(l+1)$. The general implications of this are not yet fully understood.

IV. SUMMARY AND CONCLUSIONS

In this work, we showed that a simple decoupling of the Hamilton equations of motion for the Kepler problem leads to an efficient solution for bound Keplerian orbits. The decoupling can also be used to show that the total energy is constant at each point along the orbit. Finally, we can take this form for the energy, quantize the canonical variables in it, and show that it becomes the quantum form of the Hamiltonian for systems with fixed total angular momentum.

This connection is quite unexpected. One can ask is it a general result? We do not know the answer to this. The isotropic harmonic oscillator is another exactly solvable problem for which one can use the same strategies. It is somewhat more complicated than the Kepler problem, but can be solved, yielding the orbits and the conserved energy, as well as the quantum connection; we leave the details of exploring that case to another time. It remains intriguing though to think about the connection of this classical treatment with the quantum treatment to see whether there can be new connections made about the semiclassical limit from this approach to the problem. At the very least, it does provide a nice way to relate classical mechanics solutions to quantum solutions that can ease the transition from one to the other for students studying both fields by making the two results appear more similar to each other.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

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