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Jason Tran; Leanne Doughty; James K. Freericks

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The right way to introduce complex numbers in damped harmonic oscillators

Jason Tran,[a)] Leanne Doughty,[b)] and James K. Freericks[c)]
Department of Physics, Georgetown University, 37th and O Sts. NW, Washington, District of Columbia 20057

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The introduction of complex numbers to solve the damped harmonic oscillator in classical mechanics can seem mysterious to students. We show how an approach based on Born and Jordan's 1930 quantum mechanics textbook both demystifies the use of complex numbers in classical mechanics and makes a strong connection to quantum mechanics. In that work, they converted the harmonic oscillator equations of motion into two uncoupled first-order equations. In classical mechanics, this mapping explicitly introduces complex numbers into the motion of the oscillator and directly shows how to solve for the position and momentum observables. Here, we explain how this mapping works and show how to demystify the use of complex numbers in damped-driven harmonic oscillators. As an added bonus, this approach also shows how to determine the energy as a function of time and makes a strong connection to quantum mechanics.

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I. INTRODUCTION

Instructors often teach the damped harmonic oscillator by telling students to use complex exponentials to convert the differential equation into an algebraic equation, and then "just take the real part" at the end of the calculation to determine the position because the position is a real quantity. Anecdotally, many students are troubled by that last procedure because it is physically, not mathematically argued. Here, we show a much more natural way to bring complex numbers into the analysis.

Our inspiration for this work comes from an unlikely source—the 1930 quantum textbook by Born and Jordan called *Elementare Quantenmechanik*.[1] In it, Born and Jordan showed how one can go from the Hamilton equations of motion (two coupled linear first-order differential equations) to two decoupled first-order linear differential equations. They then used this construction to factor the quantum harmonic oscillator Hamiltonian as a matrix mechanics exercise. This approach should work well with junior level students either in classical mechanics or quantum mechanics classes and can be introduced in either of them. The idea has been rediscovered by others more recently. Gauthier[2] used this idea to provide an alternative solution to the classical harmonic oscillator, which was further discussed by Tisdell,[3] while Alves[4] rediscovered the connection to quantum mechanics. Here, we focus solely on the simple harmonic oscillator. We examine undamped, damped, and damped-driven examples and analyze them in this elementary fashion, which introduces complex numbers in a natural way. Once one has the formalism down, one can discuss more properties of complex numbers in this context, as recently illustrated by Close.[5]

Experts will note that the approach given here is not new at all—it is just the standard matrix approach used to solve coupled first-order differential equations with constant coefficients.[6] However, the ideas given here go far beyond that formal approach. Indeed, it is best done with no matrices at all, as we now show.

II. COMPLEX NUMBERS AND THE CLASSICAL HARMONIC OSCILLATOR

In the U.S., nearly every introductory textbook avoids the use of complex numbers, even for problems such as the damped harmonic oscillator, for which they are natural. It does appear in a small number of introductory textbooks, such as the *Feynman Lectures in Physics*,[7] but its use is not widespread. Junior-level classical mechanics texts, such as Marion and Thorton,[8] or Taylor,[9] commonly cover it. The treatment tends to take a similar path—when confronted with the differential equation of motion for the damped-driven harmonic oscillator the idea of using a complex exponential is motivated by the fact that its derivative does not change the form of the function and thereby converts the differential equation into an algebraic equation. Then, when it comes time to determine the final motion, the student is often told by their instructor that "position is a real variable, so we must take the real part of the solution in order to obtain physical results"; textbooks usually discuss this in a more nuanced way.

In our opinion, it is much more natural to have the complex numbers arise directly within the analysis and to have the "taking of the real part" be an integral yet natural part of the analysis as we discuss next.

We first consider an undamped classical harmonic oscillator of mass m and angular frequency ω. The equations of motion are written as coupled first-order differential equations: $\dot{x} = p/m$ and $\dot{p} = F = -m\omega^2 x$. Rather than introduce complex exponentials, we instead work to decouple the equations. As one can see, each of the differential equations has the derivative of x proportional to p and the derivative of p proportional to x. Decoupling the equations means we find a single object A, whose derivative is proportional to A. This can only be accomplished when A is a linear combination of x and p. If it involves only one or the other, the derivative will not be proportional to itself, but with a properly chosen linear combination, we should be able to decouple. Hence, we choose $A = p + \alpha x$. Using the equations of motion, we immediately find that

$$\begin{aligned} \dot{A} &= \dot{p} + \alpha\dot{x} = -m\omega^2 x + \alpha\frac{p}{m} \\ &= \frac{\alpha}{m}\left(p - \frac{m^2\omega^2}{\alpha}x\right) \\ &= \lambda(p + \alpha x) = \lambda A. \end{aligned} \tag{1}$$

Since decoupling requires us to have $\dot{A} = \lambda A$ in Eq. (1), we find that $\lambda = \alpha/m$ and α is found by equating $\alpha = -m^2\omega^2/\alpha$, or

$$\alpha^2 = -m^2\omega^2, \tag{2}$$

which implies that $\alpha = \pm im\omega$. The derivative is now proportional to the original function, which is what it means to decouple the two differential equations.

The process of decoupling the equations requires us to use complex numbers in the analysis! We choose the negative root (both will be used, but we need to start somewhere) so that $A = p - im\omega x$. Then, we have $\dot{A} = -i\omega A$, which gives $A(t) = A_0 e^{-i\omega t}$, for $A_0 = p_0 - im\omega x_0$, with x_0 the initial position and p_0 the initial momentum. Next, we consider the second solution, with $\alpha = im\omega$, which satisfies $A^*(t) = p + im\omega x = A_0^* e^{i\omega t}$. Using the results for $A(t)$ and $A^*(t)$, we can directly solve for $x(t)$ and $p(t)$ as

$$x(t) = \frac{-A(t) + A^*(t)}{2im\omega} = x_0\cos\omega t + \frac{p_0}{m\omega}\sin\omega t \tag{3}$$

and

$$p(t) = \frac{A(t) + A^*(t)}{2} = p_0\cos\omega t - m\omega x_0\sin\omega t, \tag{4}$$

which are the standard solutions. One does not need to take the real part of anything! The position and momentum are simply solved for algebraically from the decoupled solutions.

The total energy,

$$\frac{p^2(t)}{2m} + \frac{1}{2}m\omega^2 x^2(t), \tag{5}$$

can also be analyzed using this approach. We first note that

$$\begin{aligned} \frac{A^*(t)A(t)}{2m} &= \frac{p^2(t)}{2m} + \frac{1}{2}m\omega^2 x^2(t) \\ &= \frac{1}{2m}|A_0|^2 = \frac{p_0^2}{2m} + \frac{1}{2}m\omega^2 x_0^2 \end{aligned} \tag{6}$$

by evaluating it once from the $A(t) = p(t) - im\omega x(t)$ and once from the form of the solution $A = A_0 e^{-i\omega t}$. Note that the energy is a constant because the product of the exponential factors in $A(t)$ and $A^*(t)$ equals 1. Hence, the total energy is conserved during the motion.

One of the key ideas students are exposed to here is that the energy can be written in a factorized form that immediately shows it is conserved. This result continues to hold for a number of different problems that can be solved exactly in quantum mechanics, but are not usually examined in a classical setting because their equations of motion cannot be solved analytically. Nevertheless, the factorization process itself, with the time derivatives of the A and A^* terms being opposite to each other is enough to guarantee energy conservation (for undamped systems). The decoupled equations, in general, take the following form:

$$\dot{A} = \lambda(t)A \quad \text{and} \quad \dot{A}^*(t) = -\lambda(t)A^*, \tag{7}$$

where the function $\lambda(t)$ might be implicitly defined in terms of the time-dependent position $x(t)$ and is always purely imaginary. The solution can be formally represented as

$$A(t) = A_0\exp\left(\int_0^t \lambda(t')dt'\right)$$

and

$$A^*(t) = A_0^*\exp\left(-\int_0^t \lambda(t')dt'\right). \tag{8}$$

The exponential factors cancel in A^*A, while the product is proportional to the energy up to an additive constant term. Hence, the argument is quite general. This can help instill in students a new perspective for these problems, especially as they move into a quantum setting where a similar result holds for the Schrödinger factorization method of solving quantum-mechanical problems.[10] We describe the quantum case in Sec. IV, which reduces to the standard operator approach used in most quantum mechanics classes for the harmonic oscillator (even if the Schrödinger factorization method is not often taught).

III. DAMPED HARMONIC OSCILLATOR

This decoupling approach can next be applied to the damped case, where we consider an additional damping force proportional to the velocity. Then, the equation of motion changes to $\dot{p} = -m\omega^2 x - \gamma\dot{x} = -m\omega^2 x - \gamma p/m$, with $\gamma > 0$. We assume that we are in the underdamped case, where $2m\omega > \gamma$. The critical and overdamped cases can be worked out following the same strategy, but we will not discuss them further here. The analysis follows just as before, because we still have a linear relationship between the derivatives of position and momentum and the position and momentum. We define $A = p + \alpha x$ again and find that

$$\dot{A} = -m\omega^2 x - \gamma\frac{p}{m} + \alpha\frac{p}{m} = \frac{\alpha - \gamma}{m}\left(p - \frac{m^2\omega^2}{\alpha - \gamma}x\right). \tag{9}$$

As before, requiring $\dot{A} = \lambda A$ and solving for α yields

$$\alpha^2 - \gamma\alpha + m^2\omega^2 = 0 \tag{10}$$

or

$$\alpha = \frac{\gamma}{2} \pm im\omega\sqrt{1 - \frac{\gamma^2}{4m^2\omega^2}}. \tag{11}$$

Finally, choosing the negative sign, we have

$$\lambda = \frac{\alpha - \gamma}{m} = -\frac{\gamma}{2m} - i\omega\sqrt{1 - \frac{\gamma^2}{4m^2\omega^2}}, \tag{12}$$

and the equations to be solved are $\dot{A} = \lambda A$ and $\dot{A}^* = \lambda^* A^*$ with constants λ and λ^*. These are solved by

$$\begin{aligned} A(t) &= A_0 e^{-(\gamma t/2m) - i\omega t\sqrt{1-(\gamma^2/4m^2\omega^2)}} \\ &= p(t) + \left(\frac{\gamma}{2} - im\omega\sqrt{1-\frac{\gamma^2}{4m^2\omega^2}}\right)x(t), \end{aligned} \tag{13}$$

with

$$A_0 = p_0 + \left(\frac{\gamma}{2} - im\omega\sqrt{1-\frac{\gamma^2}{4m^2\omega^2}}\right)x_0 \tag{14}$$

and the complex conjugate of these results for $A^*(t)$. Solving for $x(t)$ and $p(t)$ gives the usual results of

$$\begin{aligned} x(t) &= \frac{-A(t) + A^*(t)}{\alpha^* - \alpha} \\ &= \left(x_0 \cos\omega_\gamma t + \frac{p_0 + \frac{\gamma x_0}{2}}{m\omega_\gamma}\sin\omega_\gamma t\right)e^{-\gamma t/2m} \end{aligned} \tag{15}$$

and

$$\begin{aligned} &p(t) + \frac{\gamma}{2}x(t) \\ &\quad= \frac{A(t) + A^*(t)}{2} \\ &\quad= \left[\left(p_0 + \frac{\gamma x_0}{2}\right)\cos\omega_\gamma t - m\omega_\gamma x_0 \sin\omega_\gamma t\right]e^{-\gamma t/2m}, \end{aligned} \tag{16}$$

with $\omega_\gamma = \omega\sqrt{1-(\gamma^2/4m^2\omega^2)}$. Finally using Eq. (15) to subtract $\gamma x(t)/2$ from both sides of Eq. (16) gives us

$$\begin{aligned} p(t) = \Bigg[&p_0\cos\omega_\gamma t - m\omega_\gamma\left(1 + \frac{\gamma^2}{4m^2\omega_\gamma^2}\right)x_0 \sin\omega_\gamma t \\ &- \frac{\gamma}{2m\omega_\gamma}p_0 \sin\omega_\gamma t\Bigg]e^{-\gamma t/2m}, \end{aligned} \tag{17}$$

which is the standard result.

The energy analysis is more complicated here. We have $A^*(t)A(t) = |A_0|^2 e^{-\gamma t/m}$, but it is no longer proportional to just the sum of the kinetic and potential energies in Eq. (5). It has an extra term given by $\gamma x(t)p(t)$. By removing that term, one can see that the energy does decay as $e^{-\gamma t/m}$, but with complicated behavior. It is given by

$$\begin{aligned} E &= \frac{A^*(t)A(t)}{2m} - \frac{\gamma x(t)p(t)}{2m} \\ &= \left(\frac{p_0^2}{2m} + \frac{\gamma x_0 p_0}{2m} + \frac{1}{2}m\omega^2 x_0^2\right)\frac{\omega^2}{\omega_\gamma^2}e^{-\gamma t/m} \\ &\quad - \frac{\gamma^2}{4m^2\omega_\gamma^2}\left(\frac{p_0^2}{2m} + \frac{2m\omega^2 x_0 p_0}{\gamma} + \frac{1}{2}m\omega^2 x_0^2\right) \\ &\quad \times \cos(2\omega_\gamma t)e^{-\gamma t/m} \\ &\quad - \frac{\gamma}{2m\omega_\gamma}\left(\frac{p_0^2}{2m} - \frac{1}{2}m\omega^2 x_0^2\right)\sin(2\omega_\gamma t)e^{-\gamma t/m}, \end{aligned} \tag{18}$$

which can be found after some lengthy algebra.

The damped-driven oscillator can also be solved directly. The equations of motion become $\dot{A} = \lambda A + F(t)$ and its complex conjugate. This is an inhomogeneous linear first-order differential equation, solved by

$$\begin{aligned} A(t) &= A_0 e^{-\gamma t/2m - i\omega_\gamma t} \\ &\quad + e^{-\gamma t/2m - i\omega_\gamma t}\int_0^t dt' e^{\gamma t'/2m + i\omega_\gamma t'}F(t'), \end{aligned} \tag{19}$$

with $A_0 = p_0 + ((\gamma/2) - im\omega_\gamma)x_0$, just as before. One can find the position and momentum in the same way as we did previously, but without a concrete function to integrate, the expressions are rather formal and will not be written down. Even solving for the steady state this way is much more tedious than the standard way to do it, so this approach is not recommended for driven problems.

IV. TRANSITION TO QUANTUM MECHANICS

We now return to the undamped oscillator to explore the connection to quantum mechanics. As we saw previously, the energy for an undamped oscillator is given by $E = A^*(t)A(t)/2m$. To make the transition to quantum mechanics, we must elevate position and momentum to operators, which satisfy $[\hat{x}, \hat{p}] = i\hbar$. Here, a hat denotes an operator, as is usually done in quantum mechanics. Then, using the same definitions of $\hat{A}$ and $\hat{A}^\dagger$, with the dagger denoting a Hermitian conjugate instead of a complex conjugate for the operators, we find that

$$\begin{aligned} \frac{\hat{A}^\dagger\hat{A}}{2m} &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 - \frac{i}{2}\omega[\hat{p}, \hat{x}], \\ &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 - \frac{\hbar\omega}{2}. \end{aligned} \tag{20}$$

Thus, we have

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{1}{2m}\hat{A}^\dagger\hat{A} + \frac{\hbar\omega}{2}, \tag{21}$$

which we recognize as being essentially the standard result. (One simply needs to rescale to obtain the conventional ladder operators via $\hat{a} = i\hat{A}/\sqrt{2m\hbar\omega}$, and similarly for $\hat{a}^\dagger$.)

What about the damped case? Can it be related to dissipative quantum mechanics? There is no way to make that happen with a conventional Hamiltonian. In quantum mechanics, dissipation is approximately described via a master equation and cannot be discussed at a Hamiltonian level (unless one introduces complex and non-Hermitian Hamiltonians). So, there is no way to make a direct link to the dissipative quantum case using these results. It does motivate looking into the master equation to see if there are similarities there, but that is for future work.

V. CONCLUSIONS

The decoupling method just described provides a more natural way to analyze the harmonic oscillator and provides a simple way to connect quantum mechanics to traditional classical mechanics problems. It introduces complex numbers in a natural way to the analysis and never requires one to "just take the real part at the end." There is an added benefit that it eases the transition into quantum mechanics with a simple and direct connection between the classical energy and the quantum Hamiltonian for the harmonic oscillator.

These ideas can also be employed to efficiently determine Kepler orbits, which we will discuss separately. The connection to quantum mechanics continues with this example as well.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

[a]ORCID: 0000-0002-0280-3400.
[b]ORCID: 0009-0009-1886-1419.
[c]Electronic mail: james.freericks@georgetown.edu, ORCID: 0000-0002-6232-9165.

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